Goodwillie calculus and operads

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Dedicated to Trudie Ching on her birthday

Higher Chain Rule: the Faà di Bruno formula

Definition

A smooth function $f : \mathbb{R} \to \mathbb{R}$ has Taylor series, expanded at 0:

$$f(x) = \partial_0 f + \partial_1 f(x) + \frac{\partial_2 f(x,x)}{2} + \cdots + \frac{\partial_n f(x,\ldots,x)}{n!} + \ldots$$

where

$$\partial_n f(x_1,\ldots,x_n) = f^{(n)}(0) x_1 \cdots x_n.$$

Theorem (Arbogast, 1800; Faà di Bruno, 1857)

$f, g : \mathbb{R} \to \mathbb{R}$ smooth; $f(0) = g(0) = 0$:

$$\partial_n(gf) = \sum_{n=n_1+\cdots+n_k} \partial_k g(\partial_{n_1} f, \ldots, \partial_{n_k} f)$$

i.e.

$$\partial_*(gf) = \partial_*(g) \circ \partial_*(f).$$

Goodwillie Calculus: Taylor tower

Theorem (Goodwillie, 2003)

$F : \mathcal{C} \to \mathcal{D}$, functor between suitable ∞-categories, has a Taylor tower:

$$F \to \cdots \to P_n F \to P_{n-1} F \to \cdots \to P_1 F \to P_0 F$$

where $P_n F$ is the universal n-excisive approximation to F.

E.g. F is 1-excisive if it takes pushouts in $\mathcal{C}$ to pullbacks in $\mathcal{D}$.

Examples

(1) $Id : \mathcal{S}_* \to \mathcal{S}_*$ has a non-trivial Taylor tower:

$$P_1(Id)(X) \simeq \Omega^\infty \Sigma^\infty (X).$$

(2) $Id : \mathrm{Sp} \to \mathrm{Sp}$ is 1-excisive.

Goodwillie Calculus: Derivatives

Theorem (Goodwillie, 2003)

$F : \mathcal{C} \to \mathcal{D}$; functor between suitable pointed ∞-categories. Then the layer of the Taylor tower $D_nF := \mathrm{hofib}(P_nF \to P_{n-1}F)$ is given by

$$D_nF(X) \simeq \Omega^\infty \partial_n F(\Sigma^\infty X, \ldots, \Sigma^\infty X)_{h\Sigma_n}$$

for a symmetric multilinear functor $\partial_n F : \mathrm{Sp}(\mathcal{C})^n \to \mathrm{Sp}(\mathcal{D})$, the nth derivative of F.

Example

(Arone-Mahowald, 1999): $Id : \mathcal{S}_* \to \mathcal{S}_*$ has derivatives the ‘Lie operad’

$$\partial_n(Id) : \mathrm{Sp}^n \to \mathrm{Sp}; \quad (E_1, \ldots, E_n) \mapsto \mathrm{Lie}(n) \wedge E_1 \wedge \ldots \wedge E_n$$

(Kuhn, 2006; McCarthy, 2001): $\Sigma^\infty\Omega^\infty : \mathrm{Sp} \to \mathrm{Sp}$ has derivatives given by the ‘commutative cooperad’

$$\partial_n(\Sigma^\infty\Omega^\infty) : \mathrm{Sp}^n \to \mathrm{Sp}; \quad (E_1, \ldots, E_n) \mapsto \mathrm{Com}(n) \wedge E_1 \wedge \ldots \wedge E_n.$$

Theorem (C., 2010 (for Sp); Bauer et al., 2018 (for chain complexes))

$F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$: reduced functors with $\mathcal{D}$ stable. Then

$$\partial_*(GF) \simeq \partial_*(G) \circ \partial_*(F).$$

Corollary (Arone-C., 2011 (for $\mathcal{C} = \mathcal{S}_*$); not written down in general)

For any (pointed compactly-generated) ∞-category $\mathcal{C}$, the adjunction $\Sigma^\infty : \mathcal{C} \rightleftarrows \mathrm{Sp}(\mathcal{C}) : \Omega^\infty$ gives rise to a comonad

$$\Sigma^\infty\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$$

and hence a cooperad $\partial_*(\Sigma^\infty\Omega^\infty)$ with structure map

$$\partial_*(\Sigma^\infty\Omega^\infty) \to \partial_*(\Sigma^\infty\Omega^\infty\Sigma^\infty\Omega^\infty) \simeq \partial_*(\Sigma^\infty\Omega^\infty) \circ \partial_*(\Sigma^\infty\Omega^\infty)$$

Example (Arone-C., 2011)

For $\mathcal{C} = \mathcal{S}_*$: $\partial_*(\Sigma^\infty\Omega^\infty)$ is the commutative cooperad.

∞-Operads and Functor-(Co)Operads

Definition

A stable ∞-operad $\mathcal{O}$ is a Sp-enriched symmetric multicategory:

- collection of objects $\mathrm{ob}\,\mathcal{O}$;
- spectra $\mathcal{O}(c_1, \dots, c_n; d)$ for $c_1, \dots, c_n, d \in \mathrm{ob}\,\mathcal{O}$, for $n \geq 1$;
- composition/unit/symmetry maps s.t. diagrams commute;
- **also**: underlying ∞-category $\mathcal{O}_{\leq 1}$ is stable. (E.g. $\mathcal{O}_{\leq 1} = \mathrm{Sp}^{fin}$.)

We say $\mathcal{O}$ is corepresented on $\mathcal{C}$ if

$$\mathrm{ob}\,\mathcal{O} = \mathrm{ob}\,\mathcal{C}, \quad \mathcal{O}(c_1, \dots, c_n; d) \simeq \mathrm{Map}_{\mathcal{C}}(F_n(c_1, \dots, c_n), d)$$

for some $(F_n : \mathcal{C}^n \to \mathcal{C})$; in which case we have natural transformations

$$F_{n_1 + \dots + n_k} \to F_k(F_{n_1}, \dots, F_{n_k}), \quad F_1 \to Id$$

that make (F_n) into a functor-cooperad on $\mathcal{C}$. A functor-operad on $\mathcal{C}$ is a functor cooperad on $\mathcal{C}^{op}$.

Goodwillie Derivatives and Operads I

Lemma

The multilinearization of the n-fold cartesian product functor

$$\times : \mathcal{C}^n \to \mathcal{C}$$

is

$$\partial_n(\Sigma^\infty \Omega^\infty) : \mathrm{Sp}(\mathcal{C})^n \to \mathrm{Sp}(\mathcal{C})$$

Definition (Lurie, HA.6.2)

There is a functor-cooperad structure on $\partial_*(\Sigma^\infty \Omega^\infty)$ by multilinearizing the functor-cooperad structure on $\times$ given by maps of the form

$$\times(X_1, X_2, X_3, X_4, X_5) \xrightarrow{\sim} \times(\times(X_1, X_2), \times(X_3, X_4, X_5)).$$

Theorem (Heuts, 2015)

We can approximate objects in $\mathcal{C}$ via Tate $\partial_*(\Sigma^\infty \Omega^\infty)$-coalgebras.

Goodwillie Derivatives and Operads II: Koszul Duality

Lemma (Arone-C., 2011 (for $\mathcal{S}_*$); see Arone-Kankaanrinta, 1998)

For a (pointed compactly-generated) ∞-category $\mathcal{C}$, we have

$$\begin{aligned}\partial_*(Id) &\simeq \mathrm{Tot}(\partial_*(\Omega^\infty(\Sigma^\infty\Omega^\infty)^\bullet\Sigma^\infty) \\ &\simeq \mathrm{Tot}(\partial_*(\Omega^\infty)\circ\partial_*(\Sigma^\infty\Omega^\infty)^\bullet\circ\partial_*(\Sigma^\infty)) \\ &\simeq \mathrm{Cobar}(1,\partial_*(\Sigma^\infty\Omega^\infty),1)\end{aligned}$$

Conjecture (C., 2012, for operads in Sp; not written down in general)

There is a functor-operad structure on $\partial_*(Id)$ given by applying bar-cobar duality for stable ∞-operads to the functor-cooperad $\partial_*(\Sigma^\infty\Omega^\infty)$.

Examples

(1) (Arone-C., 2011) $\mathcal{C} = \mathcal{S}_*$: $\partial_*(Id) \simeq \mathrm{Lie}$
(2) (Clark, 2020) $\mathcal{C} = \mathrm{Alg}_{\mathcal{O}}$ for an operad $\mathcal{O}$ in Sp: $\partial_*(Id) \simeq \mathcal{O}$

Goodwillie Derivatives and Operads III: Day Convolution

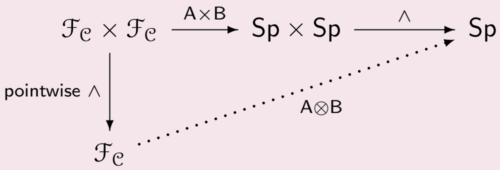
Definition (Glasman, 2016 for monoidal ∞-categories)

$\mathcal{F}_{\mathcal{C}}$: ∞-category of reduced functors $\mathcal{C} \to \mathsf{Sp}$

The Day convolution of

$$\mathsf{A}, \mathsf{B} : \mathcal{F}_{\mathcal{C}} \to \mathsf{Sp}$$

is the left Kan extension



Theorem (C., 2020)

For $X_1, \ldots, X_n \in \mathsf{Sp}(\mathcal{C})$, we have

$$\partial_n(-)(X_1, \ldots, X_n) \simeq \partial_1(-)(X_1) \otimes \cdots \otimes \partial_1(-)(X_n) : \mathcal{F}_{\mathcal{C}} \to \mathsf{Sp}.$$

Derivatives of the Identity and Day Convolution

Theorem (C., 2020)

The derivatives of the identity functor on $\mathcal{C}$ *corepresent the coendomorphism operad of* $\partial_1 : \mathcal{F}_{\mathcal{C}} \to \mathsf{Sp}$. *That is:*

$$\mathrm{Map}_{\mathsf{Sp}(\mathcal{C})}(Y, \partial_n(Id_{\mathcal{C}})(X_1, \ldots, X_n))$$

$$\simeq$$

$$\mathrm{Map}_{[\mathcal{F}_{\mathcal{C}}, \mathsf{Sp}]}(\partial_1(-)(Y), \partial_1(-)(X_1) \otimes \cdots \otimes \partial_1(-)(X_n))$$

for $X_1, \ldots, X_n, Y \in \mathsf{Sp}(\mathcal{C})$.

So we have a stable ∞*-operad* $\mathcal{I}_{\mathcal{C}}$*, given by*

$$\mathcal{I}_{\mathcal{C}}(X_1, \ldots, X_n; Y) \simeq \mathrm{Map}_{\mathsf{Sp}(\mathcal{C})}(Y, \partial_n(Id_{\mathcal{C}})(X_1, \ldots, X_n))$$

i.e. corepresented on $\mathsf{Sp}(\mathcal{C})^{op}$*, i.e.* $\partial_*(Id)$ *is a functor-operad on* $\mathsf{Sp}(\mathcal{C})$.

Derivatives of Other Functors and Day Convolution

Theorem (C., 2020)

More generally, for $F : \mathcal{C} \to \mathcal{D}$*:*

$$\mathrm{Map}_{\mathrm{Sp}(\mathcal{D})}(Y, \partial_n(F)(X_1, \ldots, X_n))$$

$$\simeq$$

$$\mathrm{Map}_{[\mathcal{F}_{\mathcal{D}}, \mathrm{Sp}]}(\partial_1(-)(Y), \partial_n(-F)(X_1, \ldots, X_n))$$

which are the terms of a $(\mathcal{I}_{\mathcal{D}}, \mathcal{I}_{\mathcal{C}})$*-bimodule* $\mathcal{M}_F$*, corepresented by the derivatives of* F*.*

Algebras over a Stable ∞-Operad

Definition

$\mathcal{O}$: (small) stable ∞-operad. An $\mathcal{O}$-algebra A in Sp consists of:

- a spectrum $A(c)$ for each $c \in \mathrm{ob}\,\mathcal{O}$;
- structure maps

$$\mathcal{O}(c_1, \ldots, c_n; d) \wedge A(c_1) \wedge \ldots \wedge A(c_n) \to A(d)$$

s.t. $A : \mathcal{O}_{\leq 1} \to \mathrm{Sp}$ is an exact functor (preserves finite (co)limits).

Denote by $\mathrm{Alg}_{\mathcal{O}}$ the ∞-category of $\mathcal{O}$-algebras in Sp.

Question

What is the stable ∞-operad $\mathcal{I}_{\mathrm{Alg}_{\mathcal{O}}}$?

Stabilization of $\mathrm{Alg}_{\mathcal{O}}$

Theorem (Basterra-Mandell, 2005)

$\mathcal{O}$*: small stable* ∞*-operad.*

$$\mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}) \simeq \mathrm{Fun}^{\mathrm{exact}}(\mathcal{O}_{\leq 1}, \mathrm{Sp}) \simeq \mathrm{Pro}(\mathcal{O}_{\leq 1})^{op}$$

where $\mathcal{O}_{\leq 1}$ *is the underlying stable* ∞*-category of* $\mathcal{O}$*.*

$\mathrm{Pro}(\mathcal{O}_{\leq 1})$ *is the* ∞*-category of pro-objects in the* ∞*-category* $\mathcal{O}_{\leq 1}$*. A cofiltered diagram* $c : \mathbb{I} \to \mathcal{O}_{\leq 1}$ *corresponds to the exact functor*

$$X \mapsto \underset{i \in \mathbb{I}}{\mathrm{colim}}\, \mathrm{Map}_{\mathcal{O}_{\leq 1}}(c(i), X)$$

Example

If $\mathcal{O} = \mathrm{Com}$, then $\mathcal{O}_{\leq 1} = \mathrm{Sp}^{fin}$ and

$$\mathrm{Sp}(\mathrm{Alg}_{\mathrm{Com}}) \simeq \mathrm{Pro}(\mathrm{Sp}^{fin})^{op} \simeq \mathrm{Sp}$$

Operad Structure on Pro-Objects

Definition

We can define a stable ∞-operad $\mathrm{Pro}(\mathcal{O})$ with underlying stable ∞-category $\mathrm{Pro}(\mathcal{O}_{\leq 1})$.

For cofiltered diagrams $c_i : \mathbb{I}_i \to \mathcal{O}_{\leq 1}$, $d : \mathbb{J} \to \mathcal{O}_{\leq 1}$, we set

$$\mathrm{Pro}(\mathcal{O})(c_1, \dots, c_n; d) := \lim_{j} \operatorname*{colim}_{(i_1,\dots,i_n)} \mathcal{O}(c_1(i_1), \dots, c_n(i_n); d(j))$$

generalizing the usual definition of morphisms of pro-objects (in case $n = 1$). Note that $\mathcal{O}$ embeds in $\mathrm{Pro}(\mathcal{O})$ as a full sub-operad (sub-multicategory).

Theorem (C., 2020)

For a small stable ∞-operad $\mathcal{O}$, we have

$$\mathcal{J}_{\mathrm{Alg}_{\mathcal{O}}} \simeq \mathrm{Pro}(\mathcal{O}).$$

Outline of Proof

Proof.

Let $\hat{\mathcal{O}}$ be the monoidal envelope of $\mathcal{O}$:

- objects: finite sequence $(c_1, \ldots, c_n)$ in $\mathcal{O}$;
- monoidal structure: concatenation.

Then there are fully faithful embeddings of stable ∞-operads

$$\mathcal{J}_{\mathrm{Alg}_{\mathcal{O}}} \hookrightarrow \mathrm{Fun}(\hat{\mathcal{O}}, \mathrm{Sp})^{Day,op} \hookleftarrow \mathrm{Pro}(\mathcal{O})$$

with the same essential image: the functors $G : \hat{\mathcal{O}} \to \mathrm{Sp}$ such that

- $G(c_1, \ldots, c_n) \simeq *$ for $n \geq 1$;
- G restricts to an exact functor $\mathcal{O}_{\leq 1} \to \mathrm{Sp}$.

$(\hookrightarrow)$: For $X \in \mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}})$:

$$\partial_1(-)(X) \mapsto (c_1, \ldots, c_n) \mapsto \partial_1(\mathrm{ev}_{c_1} \wedge \ldots \wedge \mathrm{ev}_{c_n})(X)$$

$(\hookleftarrow)$: left Kan extension along $\mathcal{O}_{\leq 1} \to \hat{\mathcal{O}}$

□

Some Further Questions

- Is there a chain rule: $\mathcal{M}_{GF} \simeq \mathcal{M}_G \circ_{\mathcal{I}_{\mathcal{C}}} \mathcal{M}_F$? [Conjecture: Yes.]

- What is the relationship between Lurie's model for $\partial_*(\Sigma^\infty \Omega^\infty)$ and $\mathcal{I}_{\mathcal{C}}$? [Bar-cobar duality for stable ∞-operads?]

- What is the relationship between the functors $\mathcal{C}\mathrm{at}_\infty \leftrightarrows \mathcal{O}\mathrm{p}_\infty$:

$$\mathcal{C} \mapsto \mathcal{I}_{\mathcal{C}}, \quad \mathcal{O} \mapsto \mathsf{Alg}_{\mathcal{O}}$$

  [Conjecture: a 'quasi-adjunction' between $(\infty, 2)$-categories.]

- How can we recover $\mathcal{C}$ (or maybe its Taylor tower à la Heuts) from $\mathcal{I}_{\mathcal{C}}$ with additional information?

  [Conjecture: resolve $\mathcal{I}_{\mathcal{C}}$ by a 'pro-operad': the coendomorphism 'pro-operad' on the ind-objects in $\mathcal{F}_{\mathcal{C}} \to \mathsf{Sp}$ of the form

$$F \mapsto [FX \to \Omega F \Sigma X \to \Omega^2 F \Sigma^2 X \to \cdots \to \partial_1(F)(X)].]$$

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