# A spectral sequence for cohomology of knot space

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# Notations

- $M$ : closed smooth manifold of dimension $\mathbf{d} \geq 4$.
- $Emb(S^1, M)$ : The space of smooth embeddings $S^1 \to M$ with $C^\infty$-topology, which we call the space of knots in $M$
  (without any base point condition).
- $\mathbf{k}$ : a fixed commutative ring (which is a PID). We do not restrict to a field of characteristic 0
- $H_*$ ($H^*$) : singular (co)homology with coefficients in $\mathbf{k}$.

# Motivation

- Recently, $Emb(S^1, M)$ is studied by Arone-Szymik, Budney-Gabai, and Kupers using Goodwillie-Weiss embedding calculus
- Motivation : construction of a computable spectral sequence (s.s.) converging to $H^*(Emb(S^1, M); \mathbf{k})$ for a simply connected $M$

# Main results

Our spectral sequence, which we call Čech spectral sequence and denote by $\check{\mathbb{E}}_r^{p\,q}$, has an algebraic presentation of $E_2$-page when

- $H^*(M)$ is a free **k**-module, and
- the Euler number $\chi(M) = 0 \in \mathbf{k}$ or $\chi(M)$ is invertible in **k** ($\chi(M) \in \mathbf{k}$ via the ring hom $\mathbb{Z} \to \mathbf{k}$)

We state main results separately into the cases of $\chi(M) = 0$ or invertible

# Poincaré algebra

## Definition 1

A Poincaré algebra $\mathcal{H}^*$ of dimension **d** is
a pair of a graded commutative algebra $\mathcal{H}^*$ and a linear isomorphism $\epsilon : \mathcal{H}^{\mathbf{d}} \to \mathbf{k}$ s. t.

$$\mathcal{H}^* \otimes \mathcal{H}^* \xrightarrow{\text{multiplication}} \mathcal{H}^* \xrightarrow{\epsilon} \mathbf{k}$$

induces a linear isomorphism $\mathcal{H}^* \cong (\mathcal{H}^{\mathbf{d}-*})^\vee$.
Let $\{a_i\}_i$ be a linear basis of $\mathcal{H}^*$ and
$(b_{ij})_{ij}$ denote the inverse of the matrix $(\epsilon(a_i \cdot a_j))_{ij}$.
$\Delta_{\mathcal{H}}$ : the diagonal class for $\mathcal{H}^*$ given by

$$\Delta_{\mathcal{H}} = \sum_{i,j} (-1)^{|a_j|} b_{ji}\, a_i \otimes a_j\, .$$

# Poincaré algebra

If $M$ is oriented, and $H^*(M)$ is a free $\mathbf{k}$-module, fixing an orientation on $M$, $H^*(M)$ is Poincaré algebra by $\epsilon : fund.class \mapsto 1 \in \mathbf{k}$.

# simplicial dg-algebra $A_{\bullet}^{\star\,*}(\mathcal{H})$

$\mathcal{H}^*$ : 1-connected (i.e. $\mathcal{H}^1 = 0$) Poincaré algebra of dim. **d**.
$e_i : \mathcal{H}^* \to (\mathcal{H}^*)^{\otimes n+1}$ : $a \mapsto 1 \otimes \cdots \otimes a \otimes \cdots \otimes 1$, insertion to $i$-th factor.

$$A_n^{\star\,*}(\mathcal{H}) := (\mathcal{H}^*)^{\otimes\, n+1} \otimes \bigwedge \{y_i, g_{ij} \mid 0 \leq \ i,j \ \leq n\}/\mathcal{I}$$

with $\deg y_i = (0, \mathbf{d}-1)$, $\quad \deg g_{ij} = (-1, \mathbf{d})$.
The ideal $\mathcal{I}$ is generated by

$$y_i^2 = g_{ij}^2 = 0, \quad g_{ii} = 0, \quad (e_i a - e_j a) g_{ij} = 0 \quad (a \in \mathcal{H}^*),$$
$$g_{ij} = (-1)^d g_{ji}, \quad g_{ij} g_{jk} + g_{jk} g_{ki} + g_{ki} g_{ij} = 0 \quad \text{(3-term relation)}$$

The differential is given by $\partial(a) = 0$ for $a \in \mathcal{H}^{\otimes n+1}$ and $\partial(g_{ij}) = f_{ij}\Delta_{\mathcal{H}}$, where
$f_{ij} : H \otimes H \to H^{\otimes n+1}$ is insertion to $i$-th and $j$-th factors.

# simplicial dg-algebra $A_\bullet^{\star\,*}(\mathcal{H})$

- The face $d_i : A_n^{\star\,*}(\mathcal{H}) \to A_{n-1}^{\star\,*}(\mathcal{H})$ $(0 \le i \le n)$ : is given by

  $$d_i(a_0 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_n & (0 \le i \le n-1) \\ \pm a_n a_0 \otimes \cdots \otimes a_{n-1} & (i = n) \end{cases}$$ and

  $$d_i(g_{j,k}) = g_{j',k'} \text{ where } j' = \begin{cases} j & (j \le i) \\ j-1 & (j > i) \end{cases}$$ , similarly for $k'$.

- the degeneracy $s_i : A_n^{\star\,*}(\mathcal{H}) \to A_{n+1}^{\star\,*}(\mathcal{H})$ : insertion of 1 to $i$-th factor and skip the index $i+1$.

# Main theorem : the case of $\chi(M) = 0$

$A_\bullet^{\star *}(\mathcal{H}) \longmapsto NA_\bullet^{\star *}(\mathcal{H})$ (normalization)

$\longmapsto H(NA_\bullet^{\star *}(\mathcal{H}))$ (homology of total complex)

**Theorem 2**

$M$ : 1-connected manifold.

Set $\mathcal{H}^* = H^*(M)$ and suppose that $\mathcal{H}^*$ is a free **k**-module and $\chi(M) = 0 \in \mathbf{k}$

$$^\exists \text{ a spec. seq. : } \check{\mathbb{E}}_2^{p\,q} \cong H(NA_\bullet^{\star *}(\mathcal{H})) \Rightarrow H^{p+q}(Emb(S^1, M)),$$

where bidegree is given by $p = *$, $q = \star - \bullet$

## Remark 3

$\check{\mathbb{E}}_2^{p\,q}$ has a graded commutative ring structure but its relation to the ring $H^*(Emb(S^1, M)$ an whether it induces ring structure on pages after $E_2$ is unclear for the speaker. It may be related to comparison of filtered ring objects in spectra and complexes

# simplicial dg-algebra $B_\bullet^{\star *}(\mathcal{H})$

$\mathcal{H}^*$ : 1-connected Poincaré algebra of dimension **d**.

Define a Poincaré algebra $S\mathcal{H}^*$ of dimension $2\mathbf{d}-1$ as follows:

$$S\mathcal{H}^* = \mathcal{H}^{\leq \mathbf{d}-2} \oplus \mathcal{H}^{\geq 2}[\mathbf{d}-1]$$
$$a \cdot \bar{b} = \overline{a \cdot b}$$

for $a \in H^{\leq \mathbf{d}-2}$, $\bar{b} \in \mathcal{H}^{\geq 2}[\mathbf{d}-1]$ corresponding to $b \in \mathcal{H}^{\geq 2}$

# simplicial dg-algebra $B^{\star\,*}_{\bullet}(\mathcal{H})$

Set

$$B^{\star\,*}_{n}(\mathcal{H}) := (S\mathcal{H}^*)^{\otimes\, n+1} \otimes \bigwedge \{h_{ij},\ g_{ij} \mid 0 \leq\ i, j\ \leq n\}/\mathcal{J}$$

with $\deg g_{ij} = (-1, \mathbf{d})$, $\deg h_{ij} = (-1, 2\mathbf{d} - 1)$. The ideal $\mathcal{J}$ is generated by

$$g_{ij}^2 = h_{ij}^2 = 0, \quad h_{ii} = g_{ii} = 0, \qquad g_{ij} = g_{ji} \quad h_{ij} = -h_{ji}$$

$$(e_i a - e_j a) g_{ij} = 0, \qquad (e_i a - e_j a) h_{ij} = 0 \quad (a \in S\mathcal{H}^*),$$

3-term relations for $g_{ij}$ and for $h_{ij}$, $\qquad (h_{ij} + h_{ki}) g_{jk} = (h_{ij} + h_{jk}) g_{ij}$

The differential is given by $\partial a = 0$ for $a \in S\mathcal{H}^{\otimes n+1}$ and
$\partial(g_{ij}) = f_{ij}\Delta_{\mathcal{H}}$, $\partial(h_{ij}) = f_{ij}\Delta_{S\mathcal{H}}$.
The face and degeneracy is similar to $A^{\star\,*}_{\bullet}(\mathcal{H})$.

# Main theorem : the case $\chi(M)$ is invertible

## Theorem 4

$M$: 1-connected manifold. Set $\mathcal{H}^* = H^*(M)$ and suppose that $\mathcal{H}^*$ is a free **k**-module and $\chi(M)$ is invertible in **k**

$$^\exists \text{ a spec. seq. : } \check{\mathbb{E}}_2^{p\,q} \cong H(NB_\bullet^{\star\,*}(\mathcal{H})) \Rightarrow H^{p+q}(Emb(S^1, M)),$$

where bidegree is given by $p = *$, $q = \star - \bullet$

We call the above spectral sequences the Čech spectral sequences.

### Remark 5

$\check{\mathbb{E}}_2^{p\,q}$ has a graded commutative ring structure but its relation to the ring $H^*(Emb(S^1, M))$ an whether it induces ring structure on pages after $E_2$ is unclear for the speaker. It may be related to comparison of filtered ring objects in spectra and complexes

# Other spectral sequences

- Vassiliev (1997) defined a s.s. converging to $H^*(LM, Emb(S^1, M))$ by discriminant method.
  - It is applicable to arbitrary manifold (including non-orientable one).
  - Its $E_2$-page has an interesting description but somewhat complicated for the speaker.
- Sinha (2009) defined a cosimplicial model for a variant of $Emb(S^1, M)$, which induces a Bousfield-Kan cohomology s.s.
  - A version of this s.s. for long knots in $\mathbb{R}^{\mathbf{d}}$ leads to the collapse of Vassiliev s.s. by Lambrechts-Turchin-Volić (2010) in $ch(\mathbf{k}) = 0$ and vanish of some differentials by de Brito-Horel (2020) in $ch(\mathbf{k}) > 0$.
  - $E_2$-page is described by cohomology of ordered configuration spaces of points in $M$ with a tangent vector, which is difficult to compute for general $M$.

# Computation for $M = S^k \times S^l$, (odd)$\times$(even)

### Corollary 6

$\mathbf{k}$ : $\mathbb{Z}$ or $\mathbb{F}_{\mathfrak{p}}$ with $\mathfrak{p}$ prime. $k$ : an odd number, $l$ : an even number with $k+5 \leq l \leq 2k-3$ and $|3k-2l| \geq 2$, or $l+5 \leq k \leq 2l-3$ and $|3l-2k| \geq 2$.
$H^* := H^*(Emb(S^1, S^k \times S^l))$.

1. We have isomorphisms $H^i = \mathbf{k}$ $(i = k-1, k, 2k-2, 2k-1, k+l)$.
2. If $\mathbf{k} = \mathbb{F}_{\mathfrak{p}}$ with $\mathfrak{p} \neq 2$, we have isomorphisms

$$H^i = \mathbf{k}^2 \; (i = k+l-2, \; k+l-1, \; 2k+l-3, \; 2k+l-2, \; 2k+l-1).$$

The inequalities ensure that differentials vanish by degree reason.

# Computation for $M = S^k \times S^l$, (even)×(even)

**Corollary 7**

Suppose $2 \in \mathbf{k}^\times$.

$k, l$ : two even numbers with $k + 2 \leq l \leq 2k - 2$ and $|3k - 2l| \geq 2$.

$H^* := H^*(Emb(S^1, S^k \times S^l))$.

We have isomorphisms

$$H^i = \mathbf{k} \quad (i = k - 1, k, l - 1, l, k + l - 3, k + l - 2, k + l - 1, 3k).$$

For any other degree $i \leq 2k + l$, $H^i = 0$. □

The inequalities ensure that differentials vanish by degree reason.

# $\pi_1(Emb(S^1, M))$ for 4-dimensional $M$

$Imm(S^1, M)$ : the space of immersions $S^1 \to M$

Question by Arone-Szymik : Is there a simp. conn. 4-dim $M$ s.t. the inclusion $i_M : Emb(S^1, M) \to Imm(S^1, M)$ has a non-trivial kernel on $\pi_1$.

(This map is always surjective.)

# $\pi_1(Emb(S^1, M))$ for 4-dimensional $M$

## Corollary 8

$M$ : simply connected, $\mathbf{d} = 4$, $H_2(M; \mathbb{Z}) \neq 0$, and
the intersection form on $H_2(M; \mathbb{F}_2)$ is represented by a matrix of which the inverse has at least one non-zero diagonal component.
Then, the inclusion $i_M$ induces an isomorphism on $\pi_1$. In particular,
$\pi_1(Emb(S^1, M)) \cong H_2(M; \mathbb{Z})$.

- For example, $M = \mathbb{C}P^2 \# \mathbb{C}P^2$ satisfies the assumption while $M = S^2 \times S^2$ does not.
- For the case $H_2(M) = 0$, by Arone-Szymik, $Emb(S^1, M)$ is simply connected.
- The case of all of the diagonal components of the matrix being zero is unclear for the speaker.

# Construction of Čech s.s.

# Sinha's cosimplicial model

- Goodwillie-Weiss embedding calculus is a framework which relates embedding spaces and configuration spaces of points in manifolds.
- Based on this, Turchin (2013) and de Brito-Weiss (2013) prove a beautiful theorem which states that that $Emb(N, M)$ is weak htpy equiv. to a space of derived maps of right modules of (framed) configuration spaces of points in $N$ or $M$.
- For knot spaces, another beautiful model which fits with Bousfield-Kan s.s. is Sinha's cosimplicial model. This is also based on the calculus.

# (co)module over an operad

- A (non-symmetric) operad is a (non-symmetric) sequence $\{\mathcal{O}(n)\}_{n\geq 1}$ with a partial composition $(-\circ_i -) : \mathcal{O}(m)\otimes\mathcal{O}(n)\to\mathcal{O}(m+n-1)$ satisfying some axioms. ( $\otimes$ : the monoidal product of the underlying monoidal category)
- A (right) $\mathcal{O}$-module is a symmetric sequence $X=\{X(n)\}_{n\geq 1}$ with a partial composition $(-\circ_i -) : X(n)\otimes\mathcal{O}(m)\to X(m+n-1)$.
- A (left) $\mathcal{O}$-comodule is a symmetric sequence $X=\{X(n)\}_{n\geq 1}$ with a partial composition $(-\circ_i -) : \mathcal{O}(m)\otimes X(m+n-1)\to X(n)$.

# little interval operad $\mathcal{D}_1$

$\mathcal{D}_1$ : the little interval operads

An element of $\mathcal{D}_1(n)$ is the $n$-tuple $\mathfrak{c} = (c_1, \dots, c_n)$ of closed intervals $c_i \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$ s. t. $c_i \cap c_j = \emptyset$ for $i \neq j$, and the labeling of $1, \dots, n$ is consistent with order of the interval $[-1/2, 1/2]$

$$(- \circ_2 -) : \mathcal{D}_1(3) \times \mathcal{D}_1(2) \to \mathcal{D}_1(4)$$

$$\mathfrak{d} \qquad \mathfrak{c} \qquad \mathfrak{e} = \mathfrak{d} \circ_2 \mathfrak{c}$$

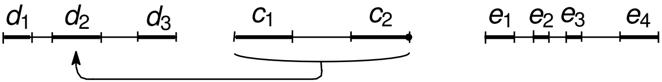



Figure: partial composition of $\mathcal{D}_1$

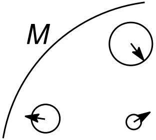
# A $\mathcal{D}_1$-module $F^M$

Fix a Riemanniann metric on $M$, $\widehat{M}$ : the tangent sphere bundle of $M$

$\delta$ : a number s.t. $0 < \delta <$the injectivity radius of $M$

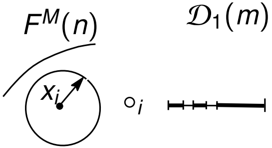
- $Ball_n(M) := \{(D_1, \dots, D_n) \mid D_i$ is a closed geodesic ball of radius $< \delta$, $D_i \cap D_j = \emptyset$ if $i \neq j\}$, topologized as a subspace of $M^n \times \mathbb{R}^n$ via (center, radius)-inclusion
- Define $F^M(n)$ as the following pullback

$$\begin{array}{ccc} F^M(n) & \longrightarrow & \widehat{M}^{\times n} \\ \downarrow & & \downarrow {\scriptstyle projection^{\times n}} \\ Ball_n(M) & \xrightarrow{center} & M^{\times n} \end{array}$$

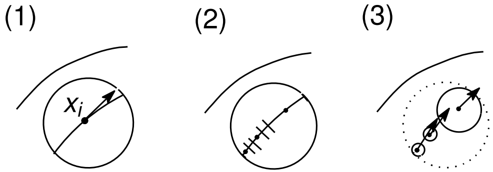


# partial composition $(- \circ_i -) \ : \ F^M(n) \times \mathcal{D}_1(m) \to F^M(m+n-1)$

The partial composition is a ”perturbed diagonal map”




is defined by



# A $\mathcal{A}_\infty$-comodule $X_A$

- $\mathcal{A}_\infty$ : the associahedral chain operad
  - generators $\{\mu_k \in \mathcal{A}_\infty(k)\}_{k\geq 2}$ ( $|\mu_k| = -k+2$ )
  - $d\mu_k = \sum_{\substack{l,p,q \\ l+p=k-1}} \pm \mu_l \circ_{p+1} \mu_q$
- For an $\mathcal{A}_\infty$-algebra $A$, Define a $\mathcal{A}_\infty$-comodule $X_A$ by
  - $X_A(n) := A^{\otimes n}$
  - $\mu_m \circ_i (a_1 \otimes \cdots a_{m+n-1}) := a_0 \otimes \cdots \otimes \mu_m(a_i, \ldots, a_{i+m-1}) \otimes \cdots \otimes a_{m+n-1}$
  - the action of $\Sigma_n$ is the standard permutation of factors.

# Hochschild complex of $\mathcal{A}_\infty$-comodule

For an $\mathcal{A}_\infty$-algebra $A$, Getzler-Jones defined a Hochschild complex $\mathbf{C}(A, A)$ as a natural generalization of that of an associative algebra.

The following lemma is a straightforward extension of Getzler-Jones.

**Lemma 9**

For a $\mathcal{A}_\infty$-comodule, $X$, there is a functorial bigraded complex $\mathrm{CH}_\bullet X$ s.t.

- For $X = X_A$, $\mathrm{CH}_\bullet X_A$ is quasi-isom. to $\mathbf{C}(A, A)$.
- $\mathrm{CH}_n X = X(n+1)$
- total degree is $* - \bullet$, where $*$ is the original cochain degree of $X(n+1)$

# from module to comodule

$\mathcal{D}_1$-module $F^M$

$\longmapsto C_*(\mathcal{D}_1)$-module $C_*(F^M)$

$\longmapsto C_*(\mathcal{D}_1)$-comodule $C^*(F^M)$

$((\alpha \circ_i f)(\sigma) = f(\sigma \circ_i \alpha)$ for $\alpha \in C_*(\mathcal{D}_1(m))$, $\sigma \in C_*(F^M(n))$, $f \in C_*(F^M(m+n-1)))$

$\longmapsto \mathcal{A}_\infty$-comodule $C^*(F^M)$.

(pulling back partial comp. by a fixed map $\mathcal{A}_\infty \to C_*(\mathcal{D}_1)$)

# Sinha spectral sequence

Filtering $\mathrm{CH}_\bullet C^*(F^M)$ by the grading $\bullet$, we have a spectral sequence $\mathbb{E}_r^{p\,q}$

**Lemma 10**

- $\mathbb{E}_r^{p\,q}$ is isom. to Bousfield-Kan cohomology s.s. associated to the (analogue of )Sinha's cosimplicial model,
- (essentially, Sinha 2009) $\mathbb{E}_r^{p\,q}$ converges to $H^*(Emb(S^1, M))$ if $M$ is simp. conn.
- $\mathbb{E}_1^{p\,q} \cong H^q(F^M(p+1))$

(Sinha considered manifolds with boundary and embeddings with some base point condition.)

# Idea of construction of Čech spectral sequence

$F^M(n)$ is htpy equiv. to $\vec{C}_n(M)$, the configuration spaces of points with tangent vector in $M$, the following pullback

$$\begin{array}{ccc} \vec{C}_n(M) & \longrightarrow & C_n(M) \\ \downarrow & & \downarrow \\ \widehat{M}^{\times n} & \longrightarrow & M^{\times n} \end{array}, \qquad C_n(M) = \{(x_1, \dots, x_n) \mid x_i \neq x_j \text{ if } i \neq j\}$$

$\Delta_{\mathrm{fat}}(M) := \cup_{p \neq q} \Delta_{p,q}(M) \subset M^{\times n}$, $\quad \Delta_{p,q}(M) = \{x_p = x_q\}$,
$\vec{\Delta}_{\mathrm{fat}}(M)$ : the space defined by the pullback

$$\begin{array}{ccc} \vec{\Delta}_{\mathrm{fat}}(M) & \longrightarrow & \Delta_{\mathrm{fat}}(M) \\ \downarrow & & \downarrow \\ \widehat{M}^{\times n} & \longrightarrow & M^{\times n} \end{array}$$

# Idea of construction of Čech spectral sequence

Idea : replace configuration spaces with fat diagonals via Poincaré-Lefschetz duality

$$C^*(\vec{C}_n(M)) \simeq C_*(\widehat{M}^{\times n}, \vec{\Delta}_{\mathrm{fat}}(M))$$

coming from $\widehat{M}^{\times n} - \vec{C}_n(M) = \vec{\Delta}_{\mathrm{fat}}(M)$ (we are loose on degree) and use Čech resolution

$$C_*(\widehat{M}^{\times n}, \vec{\Delta}_{\mathrm{fat}}(M)) \leftarrow \check{C}_{0\ n}(M) \leftarrow \check{C}_{1\ n}(M) \leftarrow \cdots$$

$$\check{C}_{k,n}(M) = \begin{cases} C_*(\widehat{M}^{\times n}) & (k = 0) \\ \oplus_I C_*(\vec{\Delta}_I M) & (k \geq 1) \end{cases}$$

where $I$ runs through set of pairs $(p, q)$ with $\# I = k$, and $\Delta_I(M) = \cap_{(p,q)\in I} \Delta_{p,q}(M)$, following Bendersky-Gitler.

# Idea of construction of Čech spectral sequence

We want to extend this to a resolution of the comodule.
Soppose we could define partial composition compatible with the differential of Čech complex

$$\begin{array}{ccccc} C_*\mathcal{D}_1(m)\otimes C^*F^M(m+n-1) & \xleftarrow{P.D.} & C_*\mathcal{D}_1(m)\otimes \check{C}_{0\ m+n-1}(M) & \longleftarrow & C_*\mathcal{D}_1(m)\otimes \check{C}_{1\ m+n-1}(M) \longleftarrow \cdots \\ \downarrow (-\circ_i-) & & \downarrow (-\circ_i-) & & \downarrow (-\circ_i-) \\ C^*(F^M(n)) & \xleftarrow{P.D.} & \check{C}_{0\ n}(M) & \longleftarrow & \check{C}_{1\ n}(M) \longleftarrow \cdots \end{array}$$

Here, $P.D$ means zigzag $C^*F^M(n) \xrightarrow{\simeq} C^*(\vec{C}_n(M)) \xrightarrow{\simeq} C_*(\widehat{M}^{\times n}, \vec{\Delta}_{\mathrm{fat}}(M)) \leftarrow \check{C}_{0,n}(M)$
(In fact, construction of partial composition is main difficulty)

# Idea of construction of Čech spectral sequence

So we would have $C_*\mathcal{D}_1$-comodule of $\check{C}^M_{*\star}$ of double complexes by $\check{C}^M_{*\star}(n) = \check{C}_{\star\, n}(M)$
$(* :$ homological, $\star$ : Čech ).

$$\longmapsto \mathrm{CH}_\bullet \check{C}^M_{*\star}$$

By filtering by $\star + \bullet$, we would get Čech s.s. $\check{\mathbb{E}}$, and
By filtering by $\bullet$, we get Sinha s.s. $\mathbb{E}$
Using this intermediate complex, we could prove convergence for simply connected $M$.

# Difficulty in construction

It is difficult (for me) to define partial compositions compatible with Čech resolution on the chain level.

This problem is analogous to construction of a chain-level intersection product which is associative, has some ”geometric description” , and makes the following diagram commutative

$$\begin{array}{ccc} C^*(M) \otimes C^*(M) & \xrightarrow{P.D.} & C_*(M) \otimes C_*(M) \\ \downarrow \cup & & \downarrow int.prod. \\ C^*(M) & \xrightarrow{P.D.} & C_*(M) \end{array}$$

A nice solution is Atiyah duality and its refinement due to R. Cohen

# Atiyah duality

Here we work in the classical homotopy category of spectra.

(Though we need some model category of spectra to justify technical issue.)

For an embedding $e : M \to \mathbb{R}^K$, $\nu$ : a tubuler nbd of $e(M)$ in $\mathbb{R}^K$.

$M^{-TM} := \Sigma^{-N} Th(\nu)$.

Different embeddings give equivalent spectra $M^{-TM}$ and equivalence can be chosen consistently. A multiplication on $M^{-TM}$:

- $\nu_\Delta$ : a tubuler neighborhood of image of $M$ in $\mathbb{R}^{2K}$ by the map

$$M \xrightarrow{\text{diagonal}} M \times M \xrightarrow{e \times e} \mathbb{R}^K \times \mathbb{R}^K$$

  taken so small that $\nu_\Delta \subset \nu \times \nu$

- multiplication $M^{-TM} \wedge M^{-TM} \to M^{-TM}$ is induced by the composition

$$\Sigma^{-N} Th(\nu) \wedge \Sigma^{-N} Th(\nu) \cong \Sigma^{-2N} Th(\nu \times \nu) \xrightarrow{\text{collapse}} \Sigma^{-2N} Th(\nu_\Delta) \cong M^{-TM}$$

# Atiyah duality

- $M^\vee$ : Spanier-Whitehead dual of $M$ with disjoint base point, i.e., $M^\vee = Map(M_+, \mathbb{S})$ ($\mathbb{S}$: sphere spectrum)
- $M^\vee$ has natural multiplication induced by pullback by $\Delta : M \to M \times M$.

**Theorem 11 (Atiyah)**

There is an equivalence of commutative ring spectrum

$$M^\vee \cong M^{-TM}$$

R. Cohen gave a refinement of this in the category of symmetric spectra. We can justify our idea using this refinement.

## Remark 12

Using the refinement of the duality, Cohen-Jones (2002) proved there is an isomorphism of graded algebra

$$(H_{*+\mathbf{d}}(LM), \text{loop product}) \cong (HH^*(C^*(M); C^*(M)), \text{cup product})$$

# dual comodule

$\mathcal{O}$: topological operad, $X$: $\mathcal{O}$-module

$\mathcal{O}$ can be considered as an operad in the category of spectra.

An $\mathcal{O}$-comodule $X^\vee$ (in spectra) is defined as follows:

- $X^\vee(n) = X(n)^\vee (= Map(X(n)_+, \mathbb{S}))$
- $(a \circ_i f)(x) = f(x \circ_i a) \qquad (a \in \mathcal{O}(m),\ f \in X^\vee(n),\ x \in X(n))$

# Key theorem

### Theorem 13 (M.)

There exists a left $\mathcal{D}_1$-comodule $\mathcal{TH}_M$ in symmetric spectra as follows.

1. There exists a zigzag of $\pi_*$-isomorphisms of left $\mathcal{D}_1$-comodules
$$(F^M)^\vee \simeq \mathcal{TH}_M .$$
2. $\mathcal{TH}_M$ has a natural Čech resolution.

There is a suitable chain functor from spectra to complexes

We can justify our idea of construction with these notions.

# Outline of proof of Cor. 8

Corollary 14 (=Cor. 8)

$M$ : simply connected, $\mathbf{d} = 4$, $H_2(M; \mathbb{Z}) \neq 0$, and the intersection form on $H_2(M; \mathbb{F}_2)$ is represented by a matrix of which the inverse has at least one non-zero diagonal component.

Then, the inclusion $i_M$ induces an isomorphism on $\pi_1$. In particular, $\pi_1(Emb(S^1, M)) \cong H_2(M; \mathbb{Z})$.

# Outline of proof of Cor. 8

- Set $H_2 = H_2(M;\mathbb{Z})$.
- By Smale-Hirsch theorem, $Imm(S^1, M) \simeq L\widehat{M}$, so $\pi_1(Imm(S^1, M)) \cong H_2$.
- $\pi_1(Emb(S^1, M))$ is finitely generated and nilpotent by a theorem for nilpotency of homotopy limits by Farjoun (2003) and the Bousfield-Kan homotopy s.s. of Sinha's model.
- It is enough to show the composition

$$Emb(S^1, M) \xrightarrow{i_M} Imm(S^1, M) \xrightarrow{cl} K(H_2, 1)$$

  induces isomorphism on $H^1(-;\mathbf{k})$ and monomorphism on $H^2(-;\mathbf{k})$ for any field $\mathbf{k}$ by a theorem of Stallings (1965). ($cl$ is the classifying map.)
- $i_M$ is induced by a map of comodules so it induces map of s.s. $\check{\mathbb{E}}_r \to E_r$ ($E_r$ is a s.s. for $L\widehat{M}$). Observing this map we have the claim on $H^1$, $H^2$.

## Remark 15

If all of the diagonal components of the inverse of intersection matrix on $H_2(M; \mathbb{F}_2)$ is zero, the map $\check{\mathbb{E}}_\infty \to E_\infty$ is not a monomorphism for $\mathbf{k} = \mathbb{F}_2$ but this does not necessarily imply the original (non-associated graded) map is not a monomorphism. So in this case, it is still unclear whether $i_M$ is an isomorphism on $\pi_1$.

# question/speculation

- Is there an essentially new element i.e. one not coming from $Imm(S^1, M)$ in $H^*(Emb(S^1, M)$ of degree higher than any given degree?
- related question : Are there any operations (e.g. multiplication) on $\check{\mathbb{E}}_r^{p\,q}$. $E_2$-page has a multiplication but it is unclear for $E_{r>2}$.
- For the case of long knots modulo immersion $\overline{Emb}_c(\mathbb{R}, \mathbb{R}^{\mathbf{d}})$, an analogue of our construction present $C^*(\overline{Emb}_c(\mathbb{R}, \mathbb{R}^{\mathbf{d}})$ as a homotopy colimit of a diagram of desuspended sphere spectra ( $\mathbf{d} \geq 4$). This may lead to a new collapse result.

# Thank you for attention!