# Moduli problems for operadic algebras

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Operad Pop-Up
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# Goal

Throughout: fix $k$ field of characteristic zero.

## Classical principle in deformation theory

Every deformation problem over $k$ is controlled by a dg-Lie algebra $\mathfrak{g}$.

## Question

Suppose that $\mathfrak{g}$ admits *additional* algebraic structure.

How can this additional structure be understood in terms of deformation problems?

# Classical example: deforming complex varieties

$X$ - proper smooth variety over $\mathbb{C}$.

Study infinitesimal deformations of $X$ along an Artin local $\mathbb{C}$-algebra $A$:

$$\begin{array}{ccc} X & \dashrightarrow & \tilde{X} \\ \downarrow & \lrcorner & \vdots \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

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- $H^1(X, T_X) \leftrightarrow$ deformations of $X$ over $\mathbb{C}[\epsilon]/\epsilon^2$.

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- $H^0(X, T_X) \leftrightarrow$ first order automorphisms of $X$.
- $H^1(X, T_X) \leftrightarrow$ deformations of $X$ over $\mathbb{C}[\epsilon]/\epsilon^2$.
- $H^2(X, T_X)$ controls *obstructions* to extending deformations:

$$\begin{array}{ccc} \tilde{X}_n & \dashrightarrow & \tilde{X}_{n+1} \\ \downarrow & \lrcorner & \vdots\ \exists \\ \mathrm{Spec}(k[\epsilon]/\epsilon^n) & \longrightarrow & \mathrm{Spec}(k[\epsilon]/\epsilon^{n+1}) \end{array} \iff \mathrm{ob}(X_n) = 0 \in H^2(X, T_X).$$

# Example: deforming complex varieties

$H^*(X, T_X)$ computed by the *Dolbeault complex*

$$\Omega^{0,*}(T_X) = \left[\Omega^{0,0}(X_{\mathbb{C}}, T_X^{1,0}) \xrightarrow{\overline{\partial}} \Omega^{0,1}(X_{\mathbb{C}}, T_X^{1,0}) \xrightarrow{\overline{\partial}} \Omega^{0,2}(X_{\mathbb{C}}, T_X) \to \dots\right].$$

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**Idea.** $\Omega^{0,*}(T_X)$ controls deformations of $X$ via the *Maurer–Cartan equation*.

More precisely, for Artin algebra $A$ with maximal ideal $\mathfrak{m}_A$

$$\left\{\text{deformations } \tilde{X} \text{ over } \mathrm{Spec}(A)\right\} \simeq \left\{ \begin{array}{c} x \in \mathfrak{m}_A \otimes \Omega^{0,1}(T_X) \\ dx + \frac{1}{2}[x, x] = 0 \end{array} \right\}$$

automorphisms

$\exp(\mathfrak{m}_A \otimes \Omega^{0,0}(T_X))$

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Higher cohomology groups: control *derived* deformations of $X$ over *dg*-Artin algebra $A$.

## Definition

An augmented commutative dg-algebra $A$ over $k$ is called *Artin* if:

- $H^*(A)$ finite-dimensional and in nonpositive degrees.
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Negative cohomology groups of a dg-Lie algebra: control *homotopies* between automorphisms.

# Formal moduli problems

## Definition

A *formal moduli problem* is a functor of $\infty$-categories

$$F : \mathrm{Art} \to \mathrm{Spaces}$$

from Artin commutative dg-algebras to spaces, such that:

- $F(k) \simeq *$.
- Schlessinger condition: for $A_1 \twoheadrightarrow A_0 \twoheadleftarrow A_2$ surjective on $H^0$:

$$F\big(A_1 \times^h_{A_0} A_2\big) \xrightarrow{\sim} F(A_1) \times^h_{F(A_0)} F(A_2)$$

'gluing along $\mathrm{Spec}(A_1) \cup_{\mathrm{Spec}(A_0)} \mathrm{Spec}(A_2)$'

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## Theorem (Pridham, Lurie)

*There is an equivalence of $\infty$-categories between formal moduli problems and dg-Lie algebras*

$$\mathrm{FMP} \xrightarrow{\simeq} \mathrm{Alg}_{\mathrm{Lie}}.$$

# Example: deforming modules

$B$ - associative algebra.
$V$ - (left) $B$-module (concentrated in cohomological degrees $\leq 0$).

Deformations of $V$ form a formal moduli problem $\mathrm{Def}_V : \mathrm{Art} \to \mathrm{Spaces}$

$$\mathrm{Def}_V(A) = \mathrm{Mod}_{A\otimes B} \times^h_{\mathrm{Mod}_B} \{V\} = \left\{ \begin{array}{c} A \otimes B\text{-modules } V_A \\ \text{with } k \otimes_A V_A \xrightarrow{\sim} V \end{array} \right\}.$$

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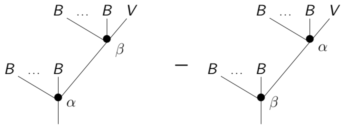
**Explicit model:** bar construction

$$\left[\mathrm{Hom}_k(V, V) \to \mathrm{Hom}(B \otimes V, V) \to \mathrm{Hom}(B \otimes B \otimes V, V) \to \dots\right]$$

with commutator bracket

$[\alpha, \beta] =$



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$B$ - associative algebra over $k$.

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$$\mathrm{Def}_B(A) = \mathrm{Alg}_A \times^h_{\mathrm{Alg}_k} \{B\} = \left\{ \begin{array}{c} A\text{-linear associative algebras } B_A \\ \text{with } k \otimes_A B_A \xrightarrow{\sim} B \end{array} \right\}.$$

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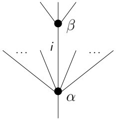
$$\overline{\mathrm{HH}}(B,B) = \left[\mathrm{Hom}(B,B) \to \mathrm{Hom}(B^{\otimes 2},B) \to \dots\right],$$

with Lie structure given by the Gerstenhaber bracket:

$$[\alpha,\beta] = \alpha\circ\beta - \beta\circ\alpha, \qquad \alpha\circ\beta = \sum_i (\pm)$$




Differential:

$$d = \Big[$$


$$, - \Big]$$

# Adding algebraic structure

## Question

Let $\mathrm{Lie} \to \mathcal{P}$ be a map of $k$-linear (dg-) operads.

If $\mathfrak{g}$ arises from a $\mathcal{P}$-algebra, what structure does the corresponding formal moduli problem have?

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{P}} & \dashrightarrow & ? \\ {\scriptstyle \mathrm{forget}} \downarrow & & \vdots \\ \mathrm{Alg}_{\mathrm{Lie}} & \xrightarrow[\sim]{} & \mathrm{FMP}. \end{array}$$

# Example 0: linear deformation problems

$\epsilon : \mathrm{Lie} \to k$ the augmentation.

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## Proposition

*Lie algebra arises as* $\Longleftrightarrow$ *corresponding formal moduli problem arises as*

$\mathfrak{g} \simeq \mathrm{triv}(V)$

$$\begin{array}{ccc} \mathrm{Art} & \longrightarrow & \mathrm{Spaces} \\ {\scriptstyle A \mapsto \mathfrak{m}_A} \big\downarrow & \overset{\simeq}{\nearrow} & \\ \mathrm{Perf}_k^{\leq 0} & & \end{array}$$

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More precisely, there is a commuting square

$$\begin{array}{ccc} \mathrm{Mod}_k & \xrightarrow{\sim} & \mathrm{FMP}_{\mathrm{lin}} \\ {\scriptstyle \mathrm{triv}}\downarrow & & \downarrow{\scriptstyle \mathrm{restrict}} \\ \mathrm{Alg}_{\mathrm{Lie}} & \xrightarrow[\sim]{} & \mathrm{FMP}. \end{array}$$

# Example 1: deforming modules

$V$ a $B$-module.

(1) The Lie algebra $\mathrm{RHom}_B(V, V)$ arises from an *associative algebra*.

(2) The corresponding formal moduli problem

$$\mathrm{Def}_V(A) = \left\{ \begin{array}{c} A \otimes B\text{-modules } V_A \\ \text{with } k \otimes_A V_A \xrightarrow{\sim} V \end{array} \right\}$$

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**In fact:** the associative extensions (1) and (2) correspond to each other via

$$\begin{array}{ccc} \mathrm{Alg}_{\mathrm{As}} & \xrightarrow{\sim} & \mathrm{FMP}_{\mathrm{As}} \\ {\scriptstyle \mathrm{forget}} \downarrow & & \downarrow {\scriptstyle \mathrm{restrict}} \\ \mathrm{Alg}_{\mathsf{Lie}} & \xrightarrow[\sim]{} & \mathrm{FMP}_{\mathrm{Com}}. \end{array}$$

# Example 2: deforming the trivial algebra

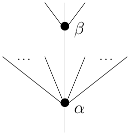
Suppose $(B, \mu = 0)$ is a *trivial* associative algebra.

Then

$$\overline{\mathrm{HH}}(B,B) = \left[\mathrm{Hom}(B,B) \xrightarrow{0} \mathrm{Hom}(B^{\otimes 2}, B) \xrightarrow{0} \dots\right]$$

together with the operation

$$\alpha \circ \beta = \sum (\pm)$$




form a *pre-Lie algebra*:

$$\alpha \circ \big(\beta \circ \gamma\big) - \big(\alpha \circ \beta\big) \circ \gamma = \alpha \circ \big(\gamma \circ \beta\big) - \big(\alpha \circ \gamma\big) \circ \beta.$$

**Question:** interpretation in terms of formal moduli problems?

# Formal moduli problems over operadic algebras

**Fix:** $\mathcal{P} \to k$ an augmented $k$-linear (symmetric, dg-) operad such that

$$H^*(\mathcal{P})(r) = 0 \qquad \text{for all } * > 0 \text{ and } r \in \mathbb{N}.$$

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## Definition

A $\mathcal{P}$-algebra $A$ is *Artin* if:

1. $H^*(A)$ is finite-dimensional and vanishes in degrees $> 0$.
2. Each $H^i(A)$ is a *nilpotent* module over the $H^0(\mathcal{P})$-algebra $H^0(A)$:

   there is some $n$ such that any $n$-fold composition of maps

   $$\mu(a_1, \ldots, a_n, -) : H^i(A) \to H^i(A) \qquad \mu \in H^0(\overline{\mathcal{P}}), \quad a_i \in H^0(A)$$

   vanishes.

**Remark.** For $\mathcal{P} = \mathrm{Com}$ the (nonunital) commutative operad:

(nonunital) Artin algebras $\Leftrightarrow$ augmentation ideals of *augmented* unital Artin algebras.

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# Formal moduli problems over operadic algebras

## Theorem (Calaque–Campos–N.)

*Let $\mathcal{P}$ be a Koszul binary quadratic operad in nonpositive cohomological degrees, with Koszul dual $\mathcal{P}^!$. Then there is an equivalence of $\infty$-categories*

$$\mathrm{MC} : \mathrm{Alg}_{\mathcal{P}^!} \xrightarrow{\sim} \mathrm{FMP}_{\mathcal{P}}$$

*between $\mathcal{P}^!$-algebras and $\mathcal{P}$-algebraic formal moduli problems.*

**Immediate examples:**

| $\mathcal{P}$ | $\mathcal{P}^!$ | |
|---|---|---|
| Com | Lie | |
| As | As | |
| $\mathrm{Pois}_n$ | $\mathrm{Pois}_n\{1-n\}$ | $(n \geq 1)$ |
| Zinb | Leib | |

# First remarks

(1) **Naturality in $\mathcal{P}$.** For every map $\mathcal{P} \to \mathcal{Q}$ of Koszul binary quadratic operads with dual $\mathcal{Q}^! \to \mathcal{P}^!$:

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{P}^!} & \xrightarrow{\sim} & \mathrm{FMP}_{\mathcal{P}} \\ {\scriptstyle \text{forget}}\downarrow & & \downarrow{\scriptstyle \text{restrict along } \mathrm{Art}_{\mathcal{Q}} \to \mathrm{Art}_{\mathcal{P}}} \\ \mathrm{Alg}_{\mathcal{Q}^!} & \xrightarrow[\sim]{} & \mathrm{FMP}_{\mathcal{Q}}. \end{array}$$

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(2) **The Maurer–Cartan equation.**
Fix $\mathfrak{g}$ a $\mathcal{P}^!$-algebra and $A \in \mathrm{Art}_{\mathcal{P}}$. Then $\mathrm{MC}_{\mathfrak{g}}(A)$ can be computed as follows:

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(2) **The Maurer–Cartan equation.**

Fix $\mathfrak{g}$ a $\mathcal{P}^!$-algebra and $A \in \mathrm{Art}_{\mathcal{P}}$. Then $\mathrm{MC}_{\mathfrak{g}}(A)$ can be computed as follows:

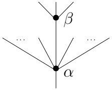
- Pick an equivalent $\mathcal{P}_\infty$-algebra $A_\infty \simeq A$ with $A_\infty$ a finite-dimensional complex.
- There is a map of operads $\mathrm{Lie}_\infty \to \mathcal{P}_\infty \otimes_{\mathrm{H}} \mathcal{P}^!$.
- Consequently, $A_\infty \otimes \mathfrak{g}$ inherits a $\mathrm{Lie}_\infty$-structure.
- The space $\mathrm{MC}_{\mathfrak{g}}(A)$ can be modeled by the simplicial set of Maurer–Cartan elements

$$\mathrm{MC}_{\mathfrak{g}}(A) \simeq \mathrm{MC}\big(A_\infty \otimes \mathfrak{g} \otimes \Omega[\Delta^\bullet]\big).$$

# Example: deforming the trivial algebra

**Recall:** for $(B, \mu = 0)$ trivial associative algebra, $\overline{\mathrm{HH}}(B, B)$ has pre-Lie structure

$$\alpha \circ \beta = \sum (\pm)$$



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### Theorem (Chapoton-Livernet)

*The pre-Lie operad is Koszul, with Koszul dual given by the permutative operad.*

A *permutative algebra* is a (nonunital) associative algebra such that

$$a(bc) = a(cb).$$

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## Proposition (informal)

*The pre-Lie algebra $\overline{\mathrm{HH}}(B, B)$ classifies a permutative formal moduli problem $\mathrm{Def}_B$.*

*For a permutative algebra $A$, the space $\mathrm{Def}_B(A)$ consists of the following deformations of $B$:*

- *a (flat) right $A$-module $\tilde{B}$, together with $\tilde{B}/\tilde{B} \cdot A \xrightarrow{\sim} B$.*
- *an associative $(A_\infty)$ product*

$$\tilde{B} \otimes_k \tilde{B} \longrightarrow \tilde{B} \cdot A \subseteq \tilde{B} \qquad \text{right } A\text{-bilinear.}$$

# About the proof

## Theorem

*For* $\mathcal{P}$ *Koszul, there is an equivalence of* $\infty$*-categories* $\mathrm{MC} : \mathrm{Alg}_{\mathcal{P}^!} \xrightarrow{\sim} \mathrm{FMP}_{\mathcal{P}}$*.*

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**Theorem**

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(1) For $(\mathcal{P}, \mathcal{P}^!)$ Koszul dual, there is an adjunction between $\infty$-categories

$$\mathfrak{D} : \mathrm{Alg}_{\mathcal{P}} \rightleftarrows \mathrm{Alg}_{\mathcal{P}^!}^{\mathrm{op}} : \mathfrak{D}'.$$

Here $\mathfrak{D}(A)$ is the linear dual of the bar construction $\mathrm{B}(A) = \big(\mathcal{P}^{\text{¡}}(A[1]), \mathrm{d}_{\mathrm{Bar}}\big)$.

(2) Define $\mathrm{MC} : \mathrm{Alg}_{\mathcal{P}^!} \longrightarrow \mathrm{FMP}_{\mathcal{P}}$ by

$$\mathrm{MC}_{\mathfrak{g}}(A) = \mathrm{Map}_{\mathrm{Alg}_{\mathcal{P}^!}}\big(\mathfrak{D}(A), \mathfrak{g}\big) \qquad A \in \mathrm{Art}_{\mathcal{P}}, \quad \mathfrak{g} \in \mathrm{Alg}_{\mathcal{P}^!}.$$

**To check:** $\mathfrak{D}$ sends pullbacks of Artin $\mathcal{P}$-algebras to pushouts of $\mathcal{P}^!$-algebras.

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**To check:** $\mathfrak{D}$ sends pullbacks of Artin $\mathcal{P}$-algebras to pushouts of $\mathcal{P}^!$-algebras.

(3) $\mathrm{MC}$ is an equivalence as soon as $\mathfrak{D}$ is fully faithful on Artin $\mathcal{P}$-algebras.

# Further generalizations

(1) For arbitrary augmented operads $\mathcal{P} \to k$: use the *bar dual* operad $\mathfrak{D}(\mathcal{P}) = (\mathrm{B}\mathcal{P})^\vee$.

Then there is an equivalence

$$\mathrm{Alg}_{\mathfrak{D}(\mathcal{P})} \xrightarrow{\sim} \mathrm{FMP}_{\mathcal{P}}$$

if the following holds:

- $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = k \cdot 1$.
- for each $n$: $\mathrm{H}^n(\mathrm{B}\mathcal{P}(r))$ vanishes for $r \gg 0$.

**Example:** $\mathcal{P} = \mathbb{E}_n$.

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**Example:** $\mathcal{P} = \mathbb{E}_n$.

(2) There is a more cumbersome condition when $\mathcal{P}(0) \neq 0$ or $\mathcal{P}(1) \neq k$.

# Further generalizations

(1) For arbitrary augmented operads $\mathcal{P} \to k$: use the *bar dual* operad $\mathfrak{D}(\mathcal{P}) = (\mathrm{B}\mathcal{P})^\vee$.

Then there is an equivalence

$$\mathrm{Alg}_{\mathfrak{D}(\mathcal{P})} \xrightarrow{\sim} \mathrm{FMP}_{\mathcal{P}}$$

if the following holds:

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(2) There is a more cumbersome condition when $\mathcal{P}(0) \neq 0$ or $\mathcal{P}(1) \neq k$.

(3) Relative/coloured case: replace $k$ by dg-algebra or dg-category $\mathbb{K}$ over $k$.

$$\text{augmented } \mathbb{K} \to \mathcal{P} \to \mathbb{K} \quad \rightsquigarrow \quad \text{(relative) dual } \mathbb{K}^{\mathrm{op}} \to \mathfrak{D}(\mathcal{P}) \to \mathbb{K}^{\mathrm{op}}.$$

**Example:** the theorem applies to $\mathcal{P} = \mathrm{SC}_n$.

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**Recall:** 1-coloured *augmented* (symmetric) operads $\leftrightarrow$ 1-coloured *nonunital* operads.

$\Rightarrow$ augmented 1-coloured operads are algebras over a coloured operad $\mathcal{O}_\Sigma$.

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## Theorem

*Augmented operads are equivalent to operadic formal moduli problems, i.e. functors*

$$F : \{\text{Artin augmented dg} - \text{operads}\} \longrightarrow \text{Spaces}.$$

# Remarks

(1) The operadic formal moduli problem classified by $\mathcal{P}$ is given by

$$\mathrm{MC}_{\mathcal{P}} : \mathrm{Art}_{\mathrm{Op}} \longrightarrow \mathrm{Spaces};\ \mathcal{N} \longmapsto \mathrm{Map}_{\mathrm{Op}^{\mathrm{aug}}}\big(\mathfrak{D}(\mathcal{N}), \mathcal{P}\big).$$

When $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = k$, this is equivalent to

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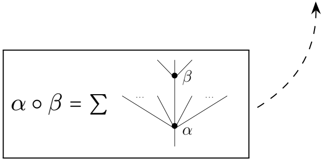
$$\begin{array}{ccccc} \mathrm{Op}^{\mathrm{aug}} & \xrightarrow{\simeq} & \mathrm{Op}^{\mathrm{nu}} & \longrightarrow & \mathrm{Alg}_{\mathrm{preLie}} \\ \mathcal{P} & \longmapsto & \overline{\mathcal{P}} & \longmapsto & \prod_r \overline{\mathcal{P}}(r)^{\Sigma_r} \end{array}$$



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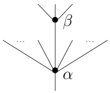

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Restricting operadic FMPs to permutative FMPs fits into

$$\begin{array}{ccc} \mathrm{Op}^{\mathrm{aug}} & \xrightarrow{\sim} & \mathrm{FMP}_{\mathrm{Op}} \\ \prod \big\downarrow & & \big\downarrow \mathrm{restrict} \\ \mathrm{Alg}_{\mathrm{preLie}} & \xrightarrow[\sim]{} & \mathrm{FMP}_{\mathrm{Perm}}. \end{array}$$

# Once more: deforming the trivial algebra

**Recall:** for $(B, \mu = 0)$, $\overline{\mathrm{HH}}(B, B)$ carries a pre-Lie structure.

**Observation.** This pre-Lie algebra arises from the (nonunital) *convolution operad*

$$\mathrm{Conv}\Big(\mathrm{coAs}\{1\}, \mathrm{End}(B)\Big)(r) = \mathrm{Hom}_k\Big(\mathrm{coAs}(r)[r-1], \mathrm{Hom}(B^{\otimes r}, B)\Big).$$

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To describe the associated formal moduli problem, we need the following:

## Definition

Given a 1-coloured operad $\mathcal{P}$, let $\mathrm{RMod}_{\mathcal{P}}^{\otimes}$ be the (big) coloured operad with:

- colours given by (cofibrant) right $\mathcal{P}(1)$-modules $V$.
- morphisms $(V_1, \ldots, V_r) \to V_0$ given by

$$V_1 \otimes \cdots \otimes V_r \to V_0 \otimes_{\mathcal{P}(1)} \mathcal{P}(r) \qquad \text{right } \mathcal{P}(1)^{\otimes r}\text{-linear.}$$

**Note:** for the unit operad $k$, all operations in $\mathrm{RMod}_k^{\otimes}$ of arity $> 1$ are zero!

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## Proposition

*The convolution operad classifies the operadic formal moduli problem*

$$\mathrm{Def}_B : \mathrm{Op}^{\mathrm{aug}} \longrightarrow \mathrm{Spaces};\ \ \mathcal{N} \longmapsto \left\{ \begin{array}{ccc} & & \mathrm{RMod}_{\mathcal{N}}^{\otimes} \\ & \nearrow & \downarrow \\ \mathbb{A}_\infty & \xrightarrow[(B,\mu=0)]{} & \mathrm{RMod}_k^{\otimes} \end{array} \right\}$$