Order and substitution on graph associahedra

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Algebra and coalgebra structures on graph associahedra

Joint with S. Forcey, *Algebraic structures on graph associahedra* arXiv:1910.00670.

Given a simple finite graph Γ, with *n* nodes, M. Carr and S. Devadoss defined a poset *Tub*(Γ), whose geometric realization is a convex polytope $\mathcal{K}\Gamma$ of dimension n-1. There exist generalizations of this construction to hypergraphs and finite CW- complexes.
The vector spaces spanned by the faces of certain families of polytopes, give another description of the Hopf algebras of faces of permutahedra and associahedra, as well as the of the free diassociative algebra on one element.
Find out some ideas to describe algebraic structures in this context.

Graph associahedra

M. Carr, S. Devadoss, *Coxeter complexes and graph associahedra*, Topol. and its Applic. 153 (1-2) (2006) 2155-2168.

Definition

A *tube* t is a set of nodes of Γ whose induced graph is a connected subgraph Γ_t of Γ.
Two tubes u and v may interact on the graph as follows:

1. Tubes are *nested* if $u \subset v$.
2. Tubes are *far apart* if $u \cup v$ is not a tube in Γ, that is, the induced subgraph of the union is not connected, (equivalently none of the nodes of u are adjacent to a node of v).

Tubes are *compatible* if they are either nested or far apart.
Γ itself the *universal tube*.

Definition

A *tubing* U of G is a set of tubes of G such that every pair of tubes in U is compatible. When Γ is connected, we assume that every tubing of G to contain (by default) its universal tube.
By the term *k-tubing* we refer to a tubing made up of k tubes, for $k \varepsilon \{1, \ldots, n\}$.

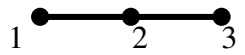
A tubing T of Γ has at most n tubes if Γ has n nodes. A tubing with k tubes is called a *k-tubing* of Γ, for $0 \leq k \leq$ n-1.

$Tub(\Gamma)$ is partially ordered by the relation:
$T \prec T'$ if T is obtained from T' by adding tubes.
M. Carr and S. Devadoss: the geometric realization of the poset $(Tub(\Gamma), \prec)$ is the barycentric division of a simple, convex polytope $\mathcal{K}\Gamma$ of dimension n-1, whose faces of dimension r are indexed by the n-r tubings of Γ, for $0 \leq r \leq n-1$.

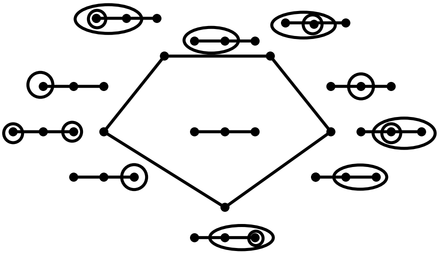
Examples

1. For the graph S_n with $\mathrm{Edg}(S_n) = \phi$, $\mathcal{K}S_n$ is the standard (n-1)-simplex
2. For the linear graph L_n with $\mathrm{Edg}(L_n) = \{(i, i+1) \mid 1 \leq i \leq n-1\}$, $\mathcal{K}L_n$ is the Stasheff polytope of dimension $n-1$, whose faces are indexed by all planar rooted trees with $n+1$ leaves.
3. For the cyclic graph C_n with $\mathrm{Edg}(C_n) = \{(i, i+1) \mid 1 \leq i \leq n-1\} \cup \{(1, n)\}$, $\mathcal{K}C_n$ is the cyclohedron of dimension $n-1$.
4. For the complete graph K_n with $\mathrm{Edg}(K_n) = \{(i, j) \mid 1 \leq i < j \leq n\}$,$\mathcal{K}K_n$ is the permutohedron of dimension $n-1$.

For $L_3 =$

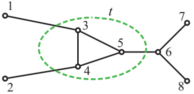


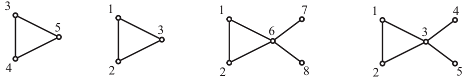
$\mathcal{K}L_3$ is given by



Reconnected graphs

Given a graph Γ and a tube t, construct a new graph Γ^*_t, called the *reconnected complement* : $\mathrm{Nod}(\Gamma^*_t) = \mathrm{Nod}(\Gamma) \setminus \{t\}$ is the set of nodes of Γ^*_t,. There is an edge between two vertices a and b of Γ^*_t if either $\{a, b\}$ to $\{a, b\} \cup t$ is connected.





Basic result The faces of dimension $n-2$ of $\mathcal{K}\Gamma$ are given by $\mathcal{K}\Gamma|_t \times \mathcal{K}\Gamma^*_t$, for a tube t in Γ.

Lemma

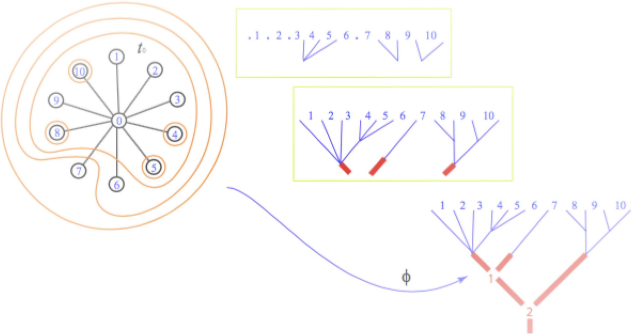
Let Γ *be a graph. For any pair of tubes* t *and* t' *in* Γ*, we have that* $(\Gamma^*_t)^*_{t'} = (\Gamma^*_{t'})^*_t$.

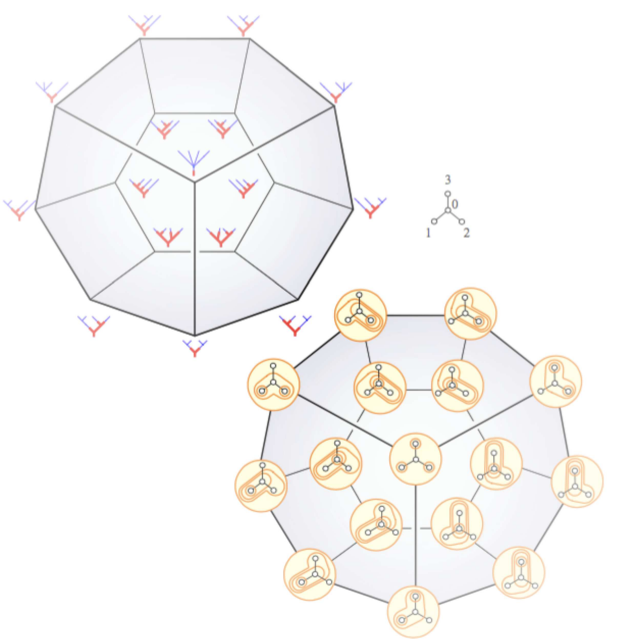
Remark When Γ is the complete graph, respectively the line graph L_n, any tube t in K_n is also complete graph K_r , respectively a line graph L_r, for some $r < n$. Moreover, the reconnected complement $(K_n)^*_{K_r} = K_{n-r}$, respectively $(L_n)^*_{L_r} = L_{n-r}$. For $\Gamma = C_n$, any tube is $t = \{i\}$ and $(C_n)^*_t = C_{n-1}$.

Suspension and colored trees

The *suspension* of Γ is the graph obained by adding a node 0 and create on edge $\{0, i\}$ for any node $i \varepsilon \Gamma$.
More generally $\Gamma \vee \Omega$ is the graph which has Γ and Ω as subgraphs, and where any node of Γ is linked to any node of Ω.

Suspension induces the notion of colored trees




3
0
1
2

PreLie coproduct

The co-preLie coproduct is defined by:

$$\Delta(T) := \sum_{t \,\varepsilon\, T} T|_{\Gamma_t} \otimes T|_{\Gamma_t^*}.$$

for $T \,\varepsilon\, \mathrm{Tub}(\Gamma)$, where we consider the empty tube $t_{\emptyset}$ and the universal tube t_Γ in order to get units.
Δ satisfies that

$$(\Delta \otimes \mathrm{Id}) \circ \Delta - (\mathrm{Id} \otimes \Delta) \circ \Delta = (\tau \otimes \mathrm{Id}) \circ ((\Delta \otimes \mathrm{Id}) \circ \Delta - (\mathrm{Id} \otimes \Delta) \circ \Delta).$$

Δ is coassociative on the vector subspace spanned by the tubings of complete graphs, it gives the coproduct on the faces of permutohedra.

Substitution

Definition

Let Γ be a graph and let T and T' be tubings of Γ, and let $t \varepsilon T$ be a tube.

1. Let S be a tubing of Γ_t which is compatible with $T|_t$. Define $T \circ_t S$ as the set of tubes s satisfying that $s \varepsilon T$, or $s \varepsilon S$ (considered as a tube in Γ).
2. Let S be a tubing of Γ_t^* which is compatible with T_t^*. A tube s in Γ belongs to the set of tubes $T \diamond S$ if it satisfies one of the following conditions
   - 2.1 $s \varepsilon T$,
   - 2.2 $s \varepsilon S$ and, as a tube in Γ, s is not linked to t,
   - 2.3 $s = s' \cup t$, for some $s' \varepsilon S$ such that s' is the union of some tubes all linked to t in Γ.

Lemma

Let T be a tubing of a graph Γ, and let t be a tube of T. We have that:

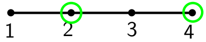
1. *if S is a tubing of Γ_t compatible with $T|_t$, then $T \circ_t S$ is a tubing of Γ,*
2. *if S is a tubing of Γ_t^* compatible with T_t^*, then $T \diamond S$ is a tubing of Γ.*

Recall that, for any tubing T of Γ, we have defined Γ_T^* as the reconnected complement of Γ after taking off all the maximal proper tubes of T. The following definition extends the notion of induced tubings to this context.

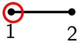
Definition

Let T be a tubing of Γ.

1. Given a tubing S of Γ^*_T, define the *substitution* of S in T to be the tubing $T \bullet_\Gamma S$ of Γ whose elements are the tubes t satisfying one of the following conditions:
   1.1 t belongs to T,
   1.2 $t = s \coprod t^{i_1} \cdots \coprod t^{i_k}$, where s is a tube of S and $\{t^{i_1}, \dots, t^{i_k}\}$ is the set of maximal proper tubes of T which are linked to at least one of a union of tubes which comprise s in Γ.
2. For any tube $t \,\varepsilon\, T$ and any tubing $S \,\varepsilon\, \mathrm{Tub}((\Gamma_t)^*_{T|_t})$, the *t-substitution* of S in T is the tubing $T \circ_t (T|_t \bullet_{\Gamma_t} S)$ on Γ. We denote it simply by $\gamma_t(T; S)$.

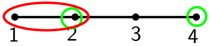


$T = \{\{2\}, \{3\}\}$

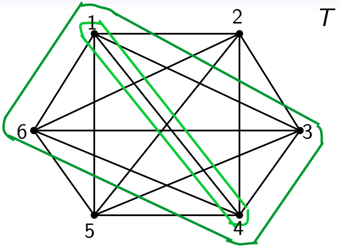


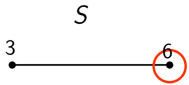
$\Gamma_T^* = L_2$ and $S = \{\{1\}\}$

we get

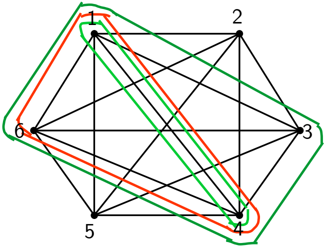


$\gamma_{t_{L_4}}(T; S)$





where $S = \{\{2\}\}$ is a tubing in $(\Gamma_{\{1,3,4,6\}})^*_{T|_{\{1,3,4,6\}}} = K_2$. We get that $\gamma_{\{\{1,3,4,6\}\}}(T; S) =$



Theorem

(Associativity of substitution) Let T be a tubing in Γ and let t be a tube in T. Given a tubing $S \varepsilon Tub((\Gamma_t)^*_{T|_t})$ and a proper tube $s \varepsilon S$, we have that:

1. *The tube s induces a tube $\tilde{s} \subsetneq t$ in Γ, given by $\tilde{s} = s \cup \{t_{i_1}, \dots, t_{i_r}\}$, where $t_{i_1}, \dots, t_{i_r}$ are the maximal proper tubes of $T|_t$ which are linked to some node of s.*
2. *The graphs $(((\Gamma_t)^*_{T|_t})_s)^*_{S|_s}$ and $(\Gamma_{\tilde{s}})^*_{\gamma_t(T,S)|_{\tilde{s}}}$ are equal.*
3. *For any tubing W of $(((\Gamma_t)^*_{T|_t})_s)^*_{S|_s}$, the tubing $\gamma_s(S;W)$ is a tubing of $(\Gamma_t)^*_{T|_t}$, which satisfies that*

$$\gamma_t(T, \gamma_s(S, W)) = \gamma_{\tilde{s}}(\gamma_t(T, S), W).$$

Note that Theorem implies that the substitution γ is associative in the following way:
Let $T = \{t_\Gamma = t^0, t^1, \dots, t^k\}$ be a tubing in Γ, and let $S^i = \{s^{0i}, \dots, s^{l_i i}\}$ be a family of tubings $S^i \varepsilon \operatorname{Tub}((\Gamma_{t^i})^*_{T|_{t^i}}$, for $0 \leq i \leq k$. Suppose that $W^{ji} \varepsilon \operatorname{Tub}(\Gamma_{s^{ji}})^*_{T|_{s^{ji}}}$ is a collection of tubings, for each pair (i,j) with $0 \leq j \leq i$. We have that:

$$\gamma(\gamma(T;S^0, \dots, S^k); W^{00}, \dots, W^{l_0 0}, \dots, W^{l_k k}) =$$
$$\gamma(T; \gamma(\gamma_{t^0}(T, S^0)|_{t^0}; W^{00}, \dots, W^{l_0 0}), \dots,).$$

Proposition Let Γ and Ω be two graphs with the same set of nodes satisfying that $\mathrm{Edg}(\Omega) \subseteq \mathrm{Edg}(\Gamma)$. For any $T \in \mathrm{Tub}(\Gamma)$, any tube $t \in T$ and any tubing $S \in \mathrm{Tub}(\Gamma^*_{T|_t})$, we have that:

$$\gamma(\mathrm{res}^{\Gamma}_{\Omega}(T); \tilde{S}^0, \dots, \tilde{S}^k) = \mathrm{res}^{\Gamma}_{\Omega}(\gamma_t(T, S)),$$

where the tube t induces a tubing $\mathrm{res}^{\Gamma}_{\Omega}(t) = \{t_1, \dots, t_k\}$, with $t_i \cap t_j = \emptyset$ for $i \neq j$, and we denote by $\tilde{S}^i$ the tubing induced by S on the reconnected complement $(\Omega_{t_i})^*_{\mathrm{res}^{\Gamma}_{\Omega}(T)|_{t_i}}$.

Proposition For any connected graph Γ, a tubing $T \in \mathrm{Tub}(\Gamma)$ may be obtained from $(\Gamma, \{t_\Gamma\})$ applying substitutions of type $\gamma_{t_\Gamma}(\ , \{t\})$, where $\{t\}$ denotes the tubing whose unique tubes are t_Γ and t.

Weak facial order

Work in progress, P. Rosero Master thesis.
Consider the extension of the weak Bruhat order to all faces of permutahedra, introduced in

1. D. Krob, M. Latapy, J.-C. Novelli, Ha-D. Phan, S. Schwer, *PseudoPermutations I: First Combinatorial and Lattice Properties*, 13th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2001), 2001.
2. P. Palacios, M. Ronco, *Weak Bruhat order on the set of faces of the permutohedron and the associahedron*, J. Algebra, 299(2) 648-678, 2006.

Studied and generalized by A. Dermenjian, C. Hohlweg, V. Pilaud, *The facial weak order and its lattice quotients*, 2016.

Let Γ be a connected simple finite graph with n nodes. The weak facial order induces a partial order on the faces of graph associahedra, given by the following conditions

1. Let $T_{\leq i}$ be the minimal tubing containing the nodes $1, \dots, i$, for some $1 \leq i < n$, then $T_{\leq i} <_f \{t_\Gamma\}$,
2. Let $T_{\geq i}$ be the minimal tubing containing the nodes $i, \dots, n$, for some $1 < i \leq n$, then $\{t_\Gamma\} <_f T_{\geq i}$,
3. If $T <_f T'$ are tubings of Γ, such that $\Gamma^*_T = \Gamma^*_{T'}$, and S is a tubing in Γ^*_T, then $T \bullet_\Gamma S <_f T' \bullet_\Gamma S$,
4. If $S <_f S'$ tubings of Γ^*_T, then $T \bullet_\Gamma S <_f T \bullet_\Gamma S'$.

In this case we get a triangulation of graph associahedra, which extends the one introduced by J.-L. Loday in *Parking functions and triangulation of the associahedron*, Proceedings of the Street's fest 2006, Contemp. Math. AMS 431 (2007), 327-340.

Graph associahedra described by operations and relations

Let Γ be a graph with $\text{Nod}(\Gamma) = [n]$, and let $\text{Nod}(t) = \{i_1 < \cdots < i_{|t|}\}$ be a tube in Γ, where $|t|$ denotes the number of nodes in t.

(a) The element $\sigma_t \,\varepsilon\, \Sigma_n$ is the permutation whose image is

$$\sigma_t := (i_1, \dots, i_{|t|}, j_1, \dots, j_{n-|t|}),$$

where $\text{Nod}(\Gamma_t^*) = [n] \setminus \text{Nod}(t) = \{j_1 < \cdots < j_{n-|t|}\}$.

(b) For any permutation $\Sigma \,\varepsilon\, \Sigma_n$, let $inv(\sigma)$ denote the number of pairs $1 \leq i < j \leq n$ satisfying that $\sigma^{-1}(j) < \sigma^{-1}(i)$ and that the pair (i, j) is an edge in Γ. The *signature* of a graph Γ is the map which assigns to any permutation $\sigma \,\varepsilon\, \Sigma_n$ the integer $(-1)^{|inv(\sigma)|}$, we denote it by $\text{sgn}^{\Gamma}(\sigma)$.

Let S be a tubing of Γ_t^* such that t is not linked to any proper tube s of S in Γ. Define the integer $\alpha(t,S)$ as follows:

- $\alpha(t,S) := -1$, when the minimal node of t is smaller than the minimal node of the disjoint union $\displaystyle\bigcup_{s\,\varepsilon\,\mathrm{Maxt}(S)} s$,
- $\alpha(t,S) := 1$, when the minimal node of t is larger than the minimal node of some maximal tube s of S.

The binary operation $\circ_{(\Gamma,t)}$ is partially defined on **Tub**, as follows:

$$S\circ_{(\Gamma,t)} W := \begin{cases} \alpha(t,W)\cdot(-1)^{|t|}\gamma_{t_\Gamma}(T_\Gamma\circ_t S, W), & \text{for } t \text{ not linked to } W, \\ \gamma_{t_\Gamma}(T_\Gamma\circ_t S, W), & \text{for } t \text{ linked to } W, \end{cases}$$

for any pair of tubings $S\,\varepsilon\,\mathrm{Tub}(\Gamma_t)$ and $W\,\varepsilon\,\mathrm{Tub}(\Gamma_t^*)$. That is, a tube $w\,\varepsilon\,T_\Gamma\circ_t S$ is either t_Γ or a tube in S.

Proposition The operations $\circ_{(\Gamma,t)}$ satisfy the following relations:

1. For two tubes t and t' in a graph Γ, which are not linked, we get that:

$$T_2 \circ_{(\Gamma,t')} (T_1 \circ_{(\Gamma^*_{t'},t)} S) = \max\{\alpha(t,S);\alpha(t',S)\}\, T_1 \circ_{(\Gamma,t)} (T_2 \circ_{(\Gamma^*_t,t')} S),$$

   for $T_1 \varepsilon \mathrm{Tub}(\Gamma_t)$, $T_2 \varepsilon \mathrm{Tub}(\Gamma_{t'})$ and $S \varepsilon \mathrm{Tub}(\Gamma^*_{t,t'})$, where $\max\{\alpha(t,S);\alpha(t',S)\}$ denotes the maximal integer between $\alpha(t,S)$ and $\alpha(t',S)$.

2. For two tubes $t' \subsetneq t$ in a graph Γ,

$$(T_2 \circ_{(\Gamma_t,t')} T_1) \circ_{(\Gamma,t)} S = T_2 \circ_{(\Gamma,t')} (T_1 \circ_{(\Gamma^*_{t'},\tilde{t})} S),$$

   for $T_1 \varepsilon \mathrm{Tub}((\Gamma_t)^*_{t'})$, $T_2 \varepsilon \mathrm{Tub}(\Gamma_{t'})$ and $S \varepsilon \mathrm{Tub}(\Gamma^*_t)$, where $\tilde{t}$ denotes the tube induced by t in $\Gamma^*_{t'}$.

Theorem

Let CGraph be the set of all graded simple connected finite graphs. The vector space **Tub***, equipped with the partially defined binary operations* $\circ_{(\Gamma,t)}$*, for any* $\Gamma \varepsilon$ *CGraph and any tube* t *in* Γ*, is the free object spanned by the set CGraph and the products* $\circ_{(\Gamma,t)}$*.*

Definition Let Γ be a finite connected simple graph, define the map $\partial : \mathbb{K}[\mathrm{Tub}(\Gamma)] \longrightarrow \mathbb{K}[\mathrm{Tub}(\Gamma)]$, , as the unique $\mathbb{K}$-linear endomorphism satisfying:

1. $\partial(T_\Gamma) = \sum_{t \subsetneq \Gamma} (-1)^{|t|} \mathrm{sgn}^\Gamma(\sigma_t)\{t\}$,
   where the sum is taken over all the tubes t is Γ different from the universal tube t_Γ.
2. For any tube t in Γ and any pair of tubings $T \varepsilon \mathrm{Tub}(\Gamma_t)$ and $S \varepsilon \mathrm{Tub}(\Gamma_t^*)$,

$$\partial(T \circ_{(\Gamma,t)} S) = \partial(T) \circ_{(\Gamma,t)} S + (-1)^{||t||} T \circ_{(\Gamma,t)} \partial(S).$$

The Theorem shows that there exists a unique linear map ∂ satisfying both conditions.

Batanin-Markl strict operadic category associated to substitution

M. Batanin, M. Markl, *Operadic categories and duoidal Deligne's conjecture*, Adv. in Math 285 (2015), 1630-1687.
R. Kaufmann, B. C. Ward *Feynman categories*, preprint arXiv 1312.1269 (2013)

Let us point out that non-symmetric operads and permutads are particular example of substitution of graph associahedra.

Operadic category associated to graph associahedra

Describe an operadic category $\mathcal{O}_{CD}$ such that the substitution of tubings provides a natural example of $\mathcal{O}_{CD}$ operad. Our model is M. Markl's operadic category Per.

Definition

Define the category $\mathcal{O}_{CD}$ as follows:

1. The objects of $\mathcal{O}_{CD}$ are pairs (Γ, T), where Γ is a connected simple finite graph and T is a tubing of Γ.
2. The homomorphisms in $\mathcal{O}_{CD}$ are given by:

$$\mathcal{O}_{CD}((\Gamma, T), (\Omega, S)) := \begin{cases} \phi\,, & \text{for } \Gamma \neq \Omega \text{ or } T \not\preceq S, \\ \{\iota_{\Gamma,T,S}\}, & \text{for } \Gamma = \Omega \text{ and } T \preceq S, \end{cases}$$

where $T \preceq S$ is Carr and Devadoss order, and means that T is obtained from S by adding compatible tubes.

Note that $\pi_0(\mathcal{O}_{CD}) = \mathrm{CGraph}$, the set of all simple connected finite graphs equipped with a total order on the set of nodes, and the terminal objects of $\mathcal{O}_{CD}$ are (Γ, T_Γ), for $\Gamma \,\varepsilon\, \mathrm{CGraph}$.

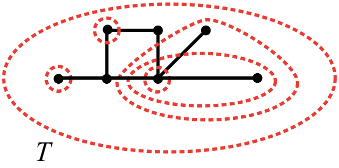
Let Γ be a finite connected simple graph and let T be a tubing of Γ, we denote by $\mathbb{L}(T)$ the number of tubes of T. The definition of a functor from the category of tubings to the category sFSet, requires a standard way to enumerate tubes in a tubing T.

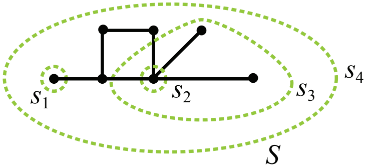
For $f = \iota_{\Gamma,T,S}$ and $i \,\varepsilon\, |(\Gamma, S)|$, there exists a unique tube $s \,\varepsilon\, S$ such that $\mathfrak{N}_S(s) = i$. Define $f^{-1}(i)$ as:

$$f^{-1}(i) = \left((\Gamma_s)^*_{(S|_s)}, (T|_s)^*_{(S|_s)}\right),$$

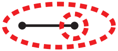
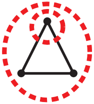
where $(\Gamma_s)^*_{(S|_s)}$ is the reconnected complement of Γ_s by the set $\mathrm{Maxt}(S|_s)$ of proper maximal tubes of $S|_s$, as described in Definition **??**, and $(T|_s)^*_{(S|_s)}$ denotes the tubing induced by $T|_s$ on $(\Gamma_s)^*_{(S|_s)}$.

For example, consider





we get that

$f^{-1}(1) =$  $= f^{-1}(2)$ $\quad f^{-1}(3) =$  $\quad f^{-1}(4) =$ 

Proposition The category $\mathcal{O}_{CD}$ is a strict operadic category .

Thanks!!!