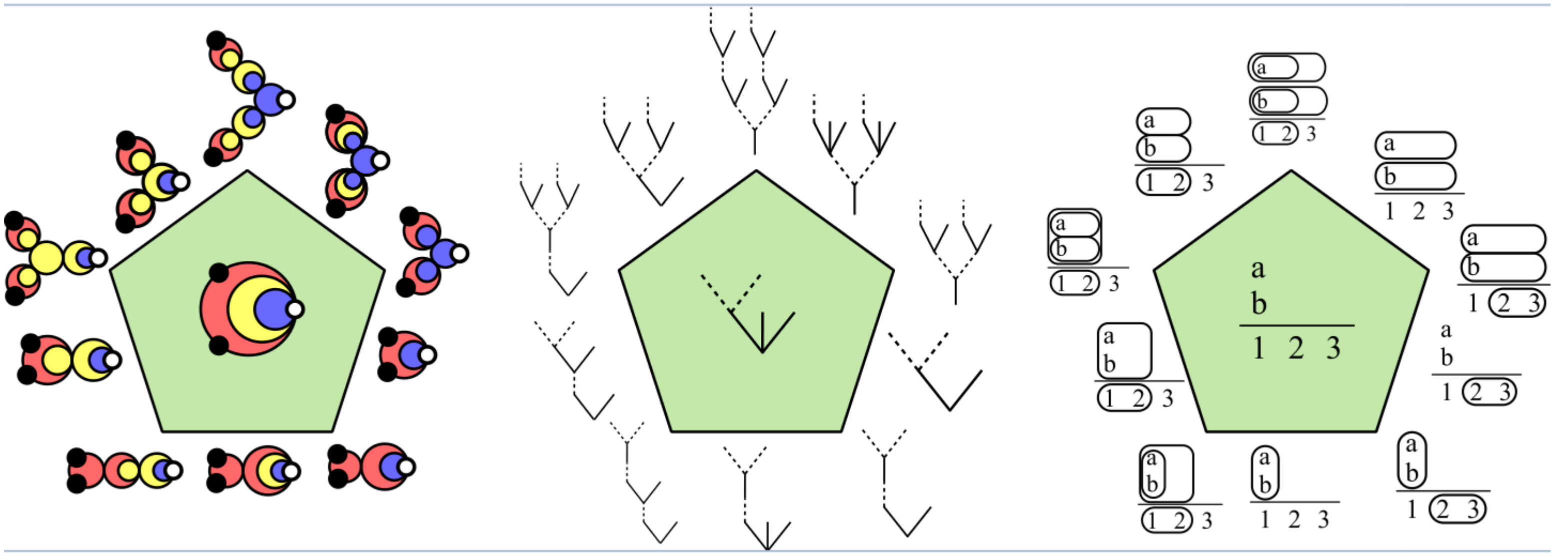
The relative 2-operad of 2-associahedra in symplectic geometry.

(parts joint with Carmeli, Oblomkov)

Plan. §1: Fuk(M) and associahedra

§2: the symplectic $(A_\infty, 2)$-category and the relative 2-operad of 2-associahedra

§3: 2-associahedra and the Fulton-MacPherson operad.



$\overline{2M}_{200}$ $\qquad$ W^{tree}_{200} $\qquad$ W^{br}_{200}

§1: the Fukaya category and associahedra.

In symplectic geometry, study:

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- symplectic manifolds: $(M^{2n}, \omega \in \Omega^2(M))$,
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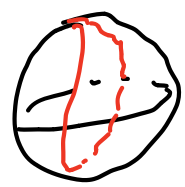
  Eg. M = real surface w/ area form,

  M = phase space of Hamiltonian dynamical system.

- Lagrangians $L^n \subset M^{2n}$, ie. submanifolds w/ $\omega|_L = 0$.

  Eg. curve $\subset$ surface,  S^2.

The Fukaya A_∞-category, $\mathrm{Fuk}(M)$.

Donaldson, Floer, Fukaya, …, mid-90s:

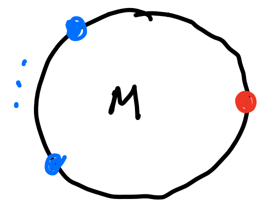
$(M, \omega) \rightsquigarrow \mathrm{Fuk}(M)$, the Fukaya A_∞-category of M.

- objects are Lagrangians $L \subset M$.
- $\forall\ d \geq 1$, have a composition operation

$$\mu_d : \hom(L_0, L_1) \otimes \cdots \otimes \hom(L_{d-1}, L_d) \longrightarrow \hom(L_0, L_d)$$

defined by "counting rigid J-holomorphic disks",

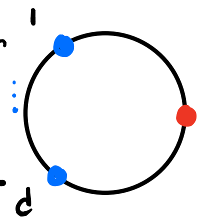
ie. certain maps w/ domains



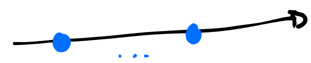
Fuk(M) and associahedra.

Fuk(M) is an A_∞-category because we define μ_d by counting maps w/ domains in $\overline{M}_d$,

$$\overline{M}_d := \overline{\left\{ \right\}}_d \Big/ PSL(2,\mathbb{R})$$



$$= \overline{\left\{ \right\}}_{\text{trans., dilation}},$$



and $(\overline{M}_d)$ is a topological realization of the associahedra!

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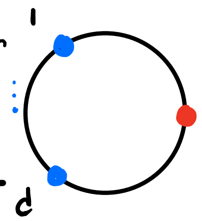
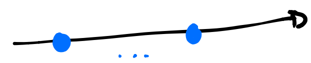


$$\left. \right\} \Big/ PSL(2,\mathbb{R}) = \overline{\left\{ \right.}$$



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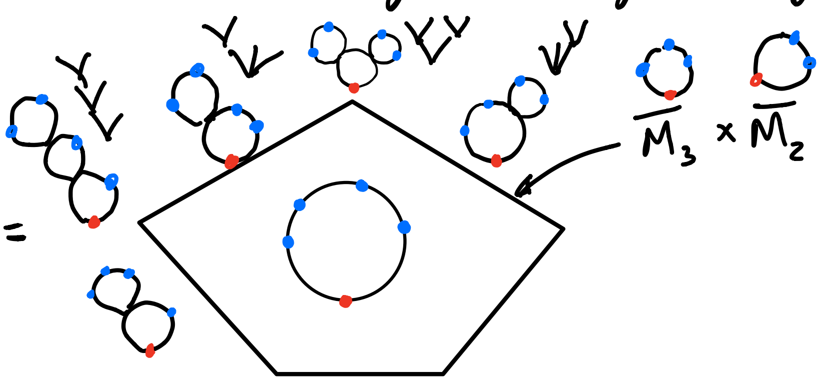
e.g.

$\overline{M}_4 =$



Fuk(M) and associahedra.

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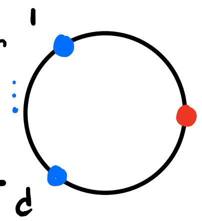
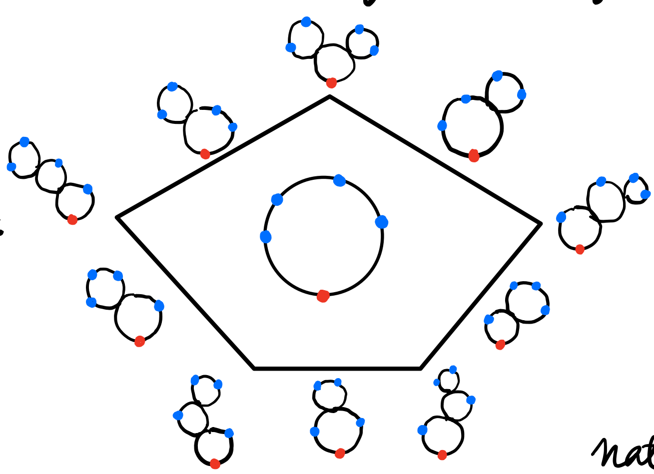
$\overline{M}_d := \overline{\Big\{$  $\Big\}}/PSL(2,\mathbb{R})$ $= \overline{\Big\{$  $\Big\}}_{\text{trans., dilation}}$,

and $(\overline{M}_d)$ is a topological realization of the associahedra!

e.g.

$\overline{M}_4 =$ 

↳ illustration of "operadic principle" in symplectic geometry: algebraic nature of invariant is inherited from operadic nature of domain spaces!

§2: the symplectic $(A_\infty, 2)$-category and 2-associahedra

Wehrheim - Woodward, ~2010:

§2: The symplectic $(A_\infty, 2)$-category and 2-associahedra

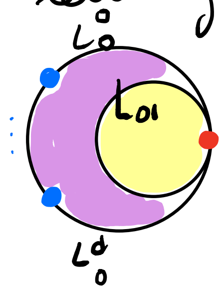
Wehrheim – Woodward, ~2010:

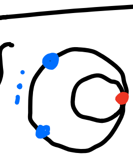
$$L_{01} \subset M_0^- \times M_1 \rightsquigarrow F_{L_{01}} : \mathrm{Fuk}(M_0) \to \mathrm{Fuk}(M_1)$$

"Lagrangian correspondence"

$$L_0 \longmapsto L_0 \circ L_{01} = L_0 \times_{M_0} L_{01} \xrightarrow{\pi_{M_1}} M_1$$

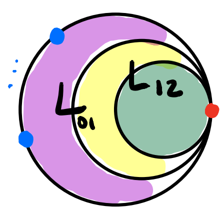
- $F_{L_{01}}$ defined on level of morphisms by counting quilted disks,



$\longleftarrow$ $\Big\{$  $\Big\}_{/\sim}$

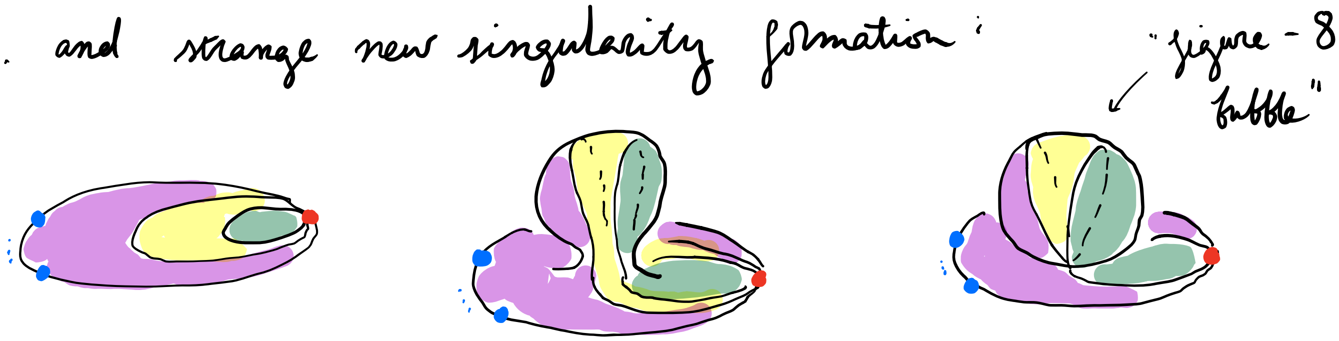
= multiplihedra

- $F_{L_{12}} \circ F_{L_{01}} \overset{A_\infty \text{ homotopy}}{\simeq} F_{L_{01} \circ L_{12}}$ via



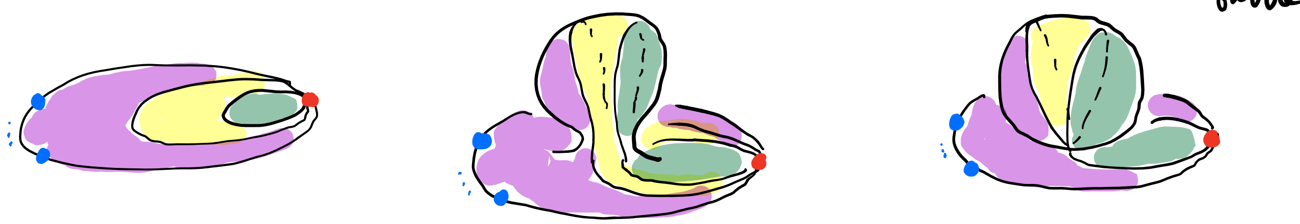
, but no higher coherences.

... and strange new singularity formation:



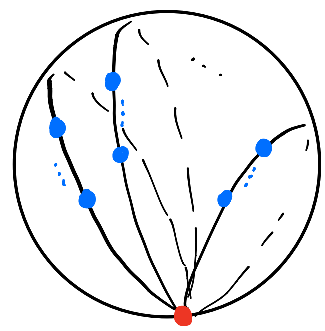
... and strange new singularity formation:

"figure-8 bubble"



⇝ Proposal (B, 2015-): Correct vehicle for functoriality of Fuk is the symplectic $(A_\infty, 2)$-category, Symp.

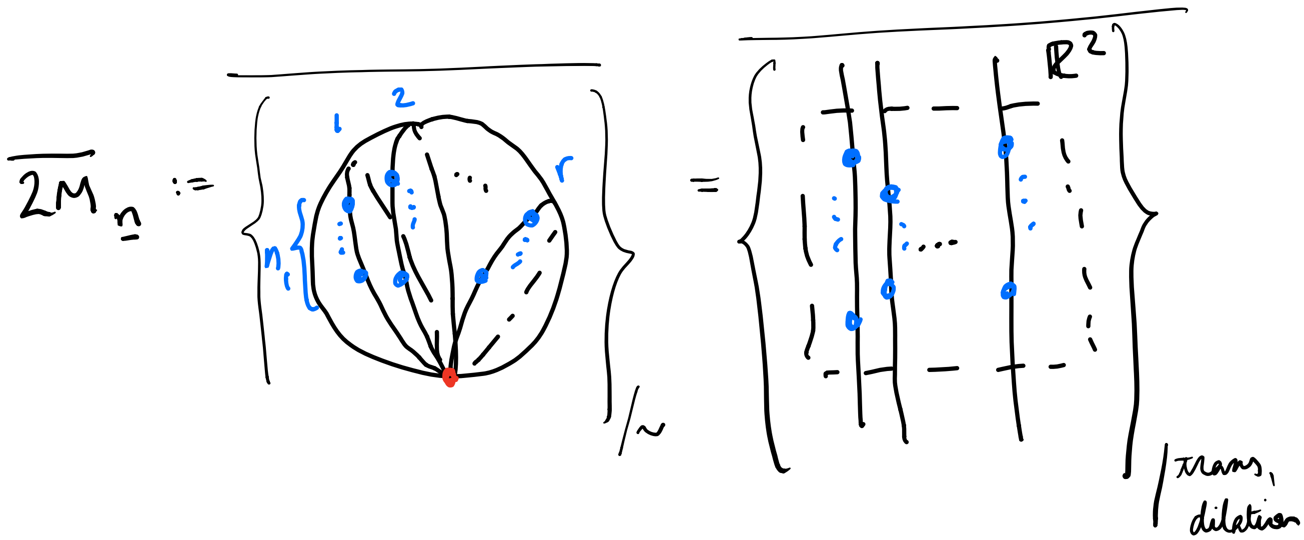
- objects are (M,ω)'s.
- $\hom(M_0, M_1) := \mathrm{Fuk}(M_0^- \times M_1)$.
- define composition of 2-morphisms by counting witch balls, i.e. maps whose domains are these quilted spheres



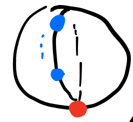
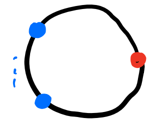
As with Fuk(M), the algebraic structure of Symp comes from the moduli spaces of domains.

(realizations of) 2-associahedra

$\forall\ r \geq 1,\ \underline{n} \in \mathbb{Z}_{\geq 0}^{r} \setminus \{\underline{0}\}$, define $\overline{2M}_{\underline{n}}$ like so:



We'll need to understand its operadic structure.

Observation : · when $r=1$, $\overline{2M}_{n_1} = \{$  $\}/\sim \;\simeq\; \{$  $\}/\sim$

$\Rightarrow \overline{2M}_{n_1}$ realizes the (n_1-2)-dim'l associahedron. $= \overline{M}_{n_1}$

(hom(M_0, M_1) is the A_∞-category $\mathrm{Fuk}(M_0^- \times M_1)$)

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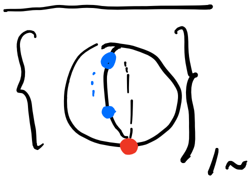
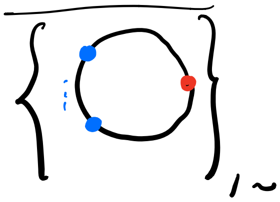
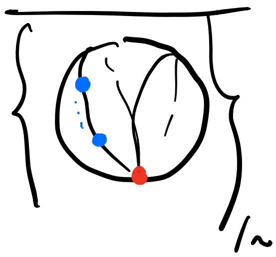
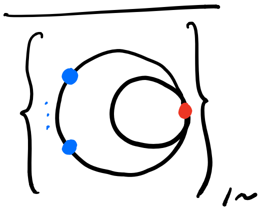
- when $r=1$, $\overline{2M}_{n_1} =$  $\simeq$ 

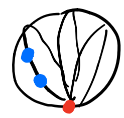
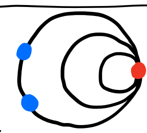
$\Rightarrow$ $\overline{2M}_{n_1}$ realizes the (n_1-2)-dim'l associahedron $= \overline{M}_{n_1}$.

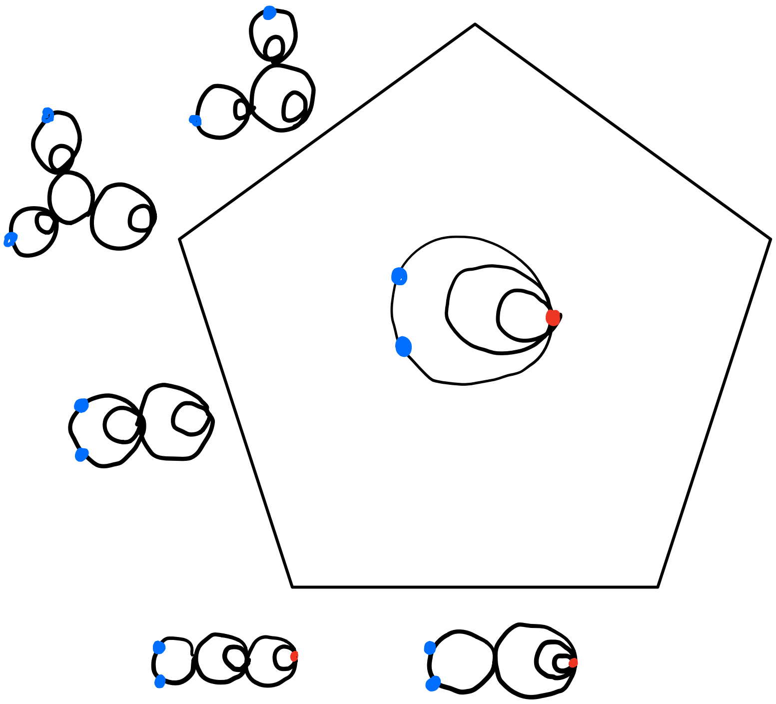
(hom(M_0, M_1) is the A_∞-category Fuk$(M_0^- \times M_1)$)

- $\overline{2M}_{(n_1,0)} =$  $\simeq$  $=$ realization of (n_1-1)-dim'l multiplihedron.

(Symp includes the functors $F_{L_{01}}$: Fuk$(M_0) \to$ Fuk(M_1))

Example: $\overline{2M}_{200}$.

$\overline{2M}_{200} = \{$  $\}/\sim$ $= \{$  $\}/\sim$.



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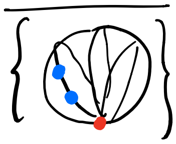
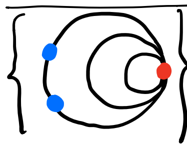
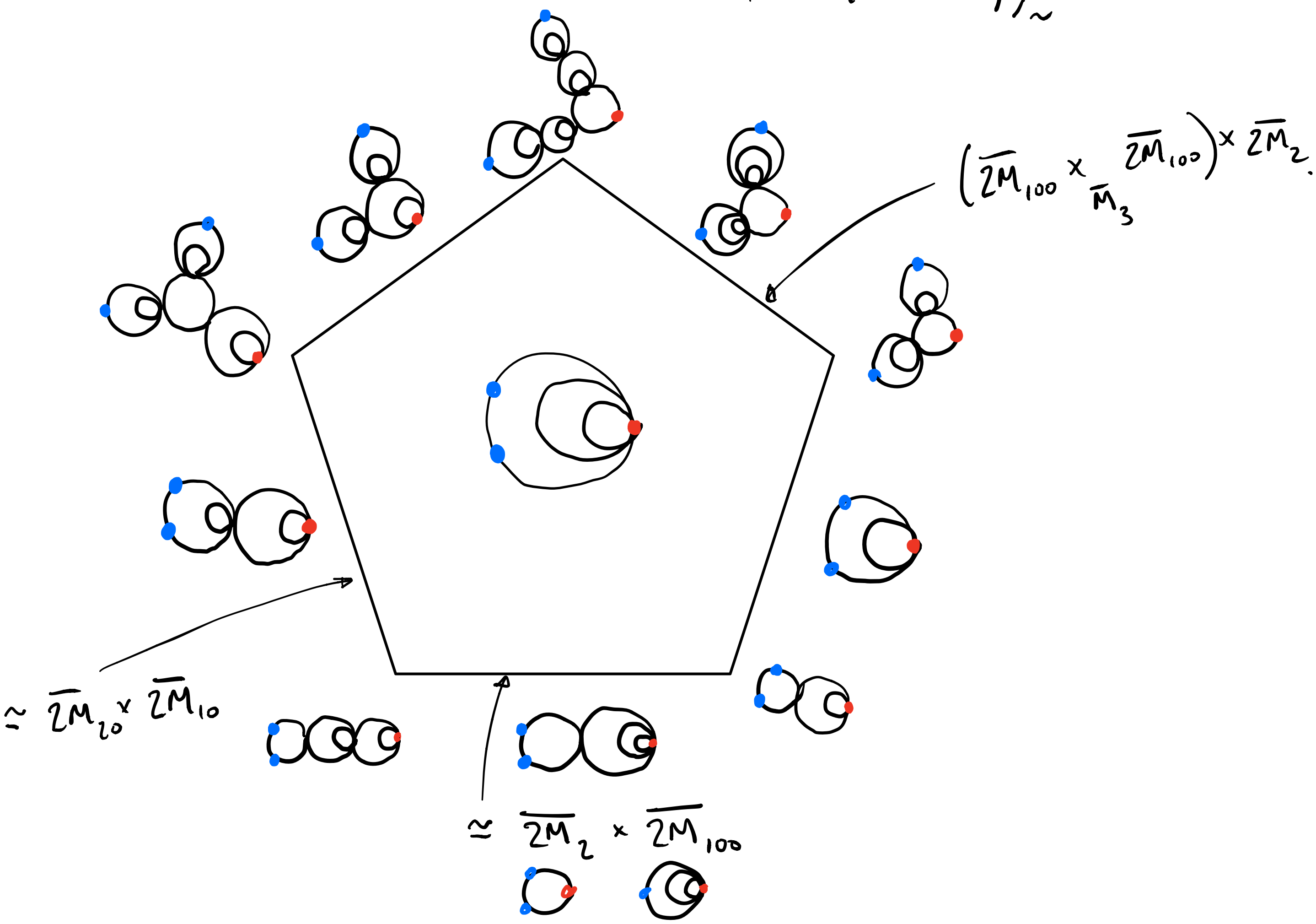
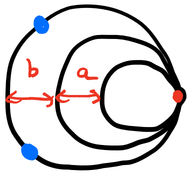
$\overline{2M}_{200} = \{$  $\}/\sim \; = \{$  $\}/\sim$.



Why was that one edge special?

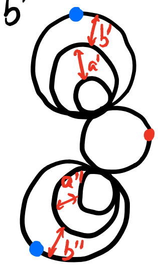


$\longrightarrow$

a'/b'

$\|$

a''/b''

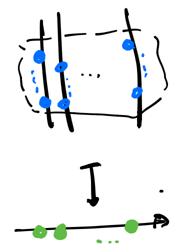


a/b preserved by automorphisms

$\rightsquigarrow$ that edge is $\simeq$ to $\left(\overline{ZM}_{100} \times_{\overline{M}_3} \overline{ZM}_{100}\right) \times \overline{ZM}_2$.

$\rightsquigarrow$ Thm (B '17, B '17): $\overline{2M}_{\underline{n}}$ is a stratified, compact, metrizable space whose poset of strata $W_{\underline{n}}$ is an abstract polytope.

- there is a forgetful map $\begin{array}{c}\overline{2M}_{\underline{n}}\\ \downarrow \\ \overline{M}_r\end{array}$ $\left(\text{and } \begin{array}{c}W_{\underline{n}}\\ \downarrow \\ K_r\end{array}\right)$,



- closed strata decompose as products of fiber products:

$$\underbrace{\overline{2M}_{\underline{n}}}_{\text{length } r} \times \prod_{1 \le i \le r} \; \prod^{\overline{M}_{s_i}}_{1 \le j \le k_i} \overline{2M}_{\underline{m}_{ij}} \hookrightarrow \overline{2M}_{\left(\sum_j \underline{m}_{1j}, \ldots, \sum_j \underline{m}_{rj}\right)}$$

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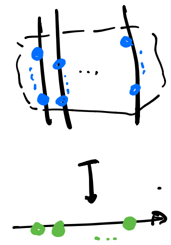
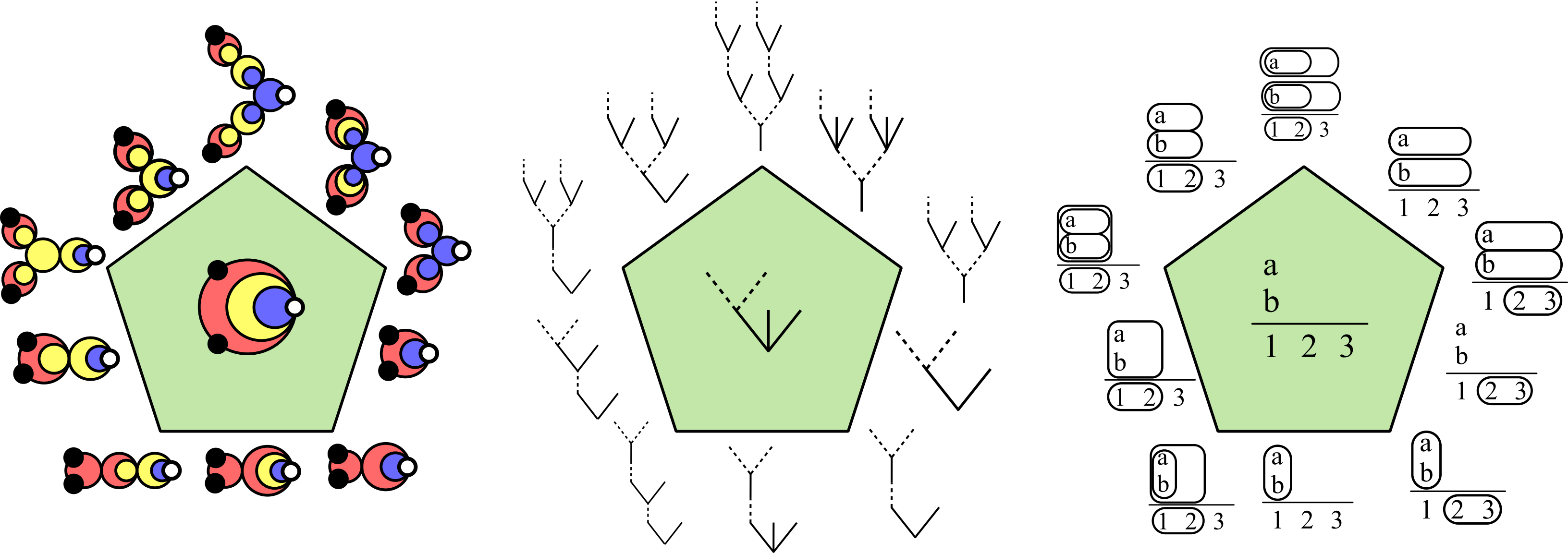


- there is a forgetful map $\begin{matrix}\overline{2M}_{n} \\ \downarrow \\ \overline{M}_r\end{matrix}$ $\left(\text{and } \begin{matrix}W_{\underline{n}} \\ \downarrow \\ K_r\end{matrix}\right)$,
- closed strata decompose as products of fiber products:

$$\overline{2M}_{\underset{\text{length } r}{\underline{n}}} \times \prod_{1\le i\le r} \prod^{\overline{M}_{s_i}}_{1\le j\le k_i} \overline{2M}_{\underline{m}_{ij}} \hookrightarrow \overline{2M}_{\left(\sum_j \underline{m}_{1j}, \dots, \sum_j \underline{m}_{rj}\right)}$$

Thm (B - Oblomkov, '19): $\overline{2M}_{\underline{n}}$ is an $(|\underline{n}|+r-3)$-dim'l manifold w/ generalized corners

$$\Longrightarrow \overline{2M}_{\underline{n}} \cong \overline{B}^{|\underline{n}|+r-3}.$$


a
b
1 2 3

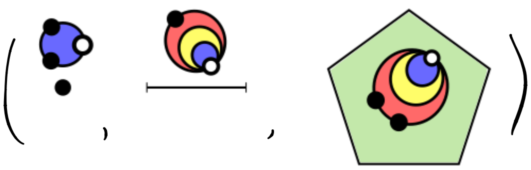
$\overline{ZM}_{200}$ W_{200}^{tree} W_{200}^{br}

Operadic structure of 2-associahedra.

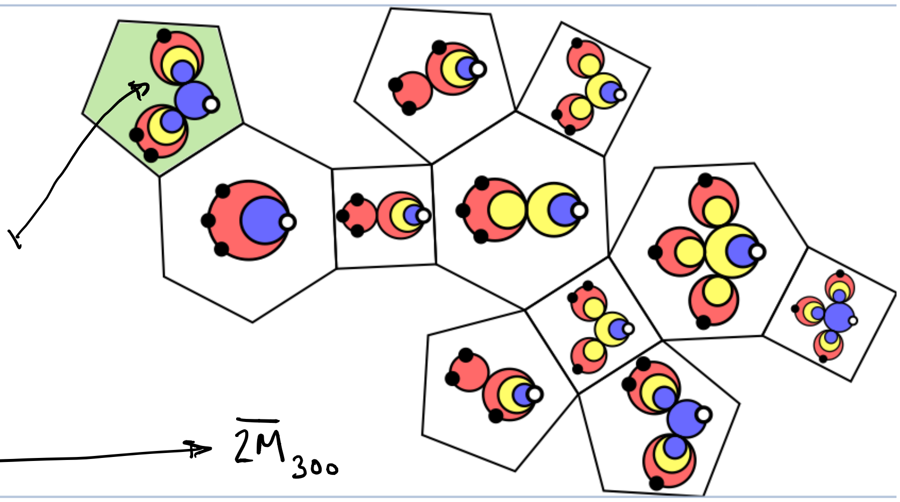
Recall operadic structure of associahedra ($\overline{M}_r$):

$\overline{M}_3 \times \dot{\overline{M}}_2 \hookrightarrow \overline{M}_4$.

Not so simple for 2-associahedra!



$\overline{2M}_2 \times \left(\overline{2M}_{100} \times_{\overline{M}_3} \overline{2M}_{200} \right) \hookrightarrow \overline{2M}_{300}$



Relative 2-operads

So $(\overline{2M}_{\underline{n}})$ is not an operad, but a 2-operad relative to $(\overline{M}_r)$.

Def (B - Carmeli, '18). A non-Σ relative 2-operad in a category $\mathcal{C}$ w/ finite limits is a pair

a non-Σ operad in $\mathcal{C}$

$$\left((P_r)_{r \geq 1},\ (Q_{\underline{m}})_{\substack{\underline{m} \in \mathbb{Z}^r_{\geq 0} \setminus \{\underline{0}\},\\ r \geq 1}} \right)$$

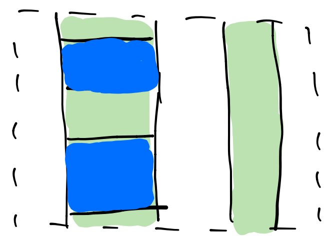
equipped with:

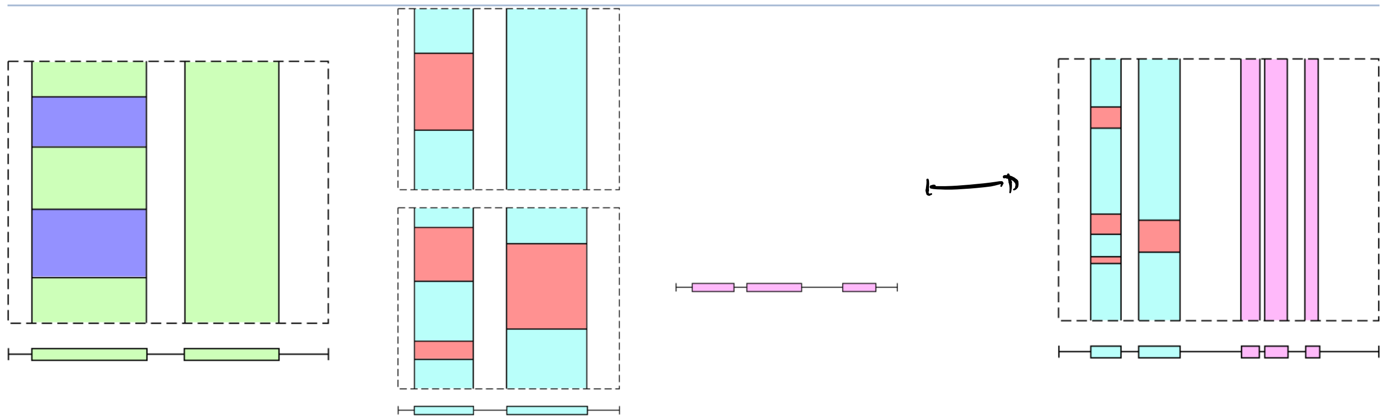
- projections $Q_{\underline{m}} \longrightarrow P_r$;
- compositions $Q_{\underline{n}} \times \prod_{1 \leq i \leq r} \underbrace{\prod_{1 \leq j \leq k_i} Q_{\underline{m}_{ij}}}_{P_{s_i}} \longrightarrow Q_{(\sum_j \underline{m}_{1j}, \dots, \sum_j \underline{m}_{rj})}$

satisfying suitable coherences.

a 2^{nd} example of a relative 2-operad: "little squares in little strips".

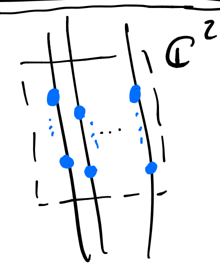
- (P_r) := little intervals operad, eg. $P_3 = \{$  $\}$.
- $Q_{\underline{n}}$:= configuration space of r vertical strips in $[0,1]^2$, where the i-th strip has n_i squares inside it.

Eg, $Q_{20} =$  .



$$Q_{20} \times \left(Q_{10} \times_{P_2} Q_{21}\right) \times P_3 \longrightarrow Q_{31000}$$

3rd example (B - Oblomkov, '19): Complex version of 2-associahedra.

$\overline{2M}_{\underline{n}}(\mathbb{C}) := \Bigg\{$  $\Bigg\}\Big/\sim$ ← includes

$\boxed{\text{Categories over relative 2-operads.}}$

Can define a category over a relative 2-operad.

$(A_\infty, 2)$-category $:=$ category over $\left((\bar{M}_r), (\overline{2M}_{\underline{n}}) \right)$.

Categories over relative 2-operads.

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$(A_\infty, 2)$ - category := category over $((\bar{M}_r), (\overline{2M}_{\underline{n}}))$.

Q. A_∞- categories form a model for stable $(\infty,1)$ - categories.
Do $(A_\infty, 2)$ - categories form a model for some kind of $(\infty, 2)$ - categories?

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Remark. Batanin has a notion of n - operads. A 2-operad is a relative 2-operad over Ass, and relative 2-operads can be similarly interpreted as 2-operads.

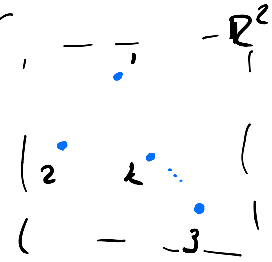
§3: 2-associahedra and the Fulton-MacPherson operad.

$(\infty, 2)$-categories : $(A_\infty, 2)$-categories :: algebra over little 2-disks : ???

$\rightsquigarrow$ expect connection between 2-associahedra and little 2-disks.

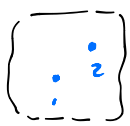
Recall Fulton–MacPherson operad (FM_k):

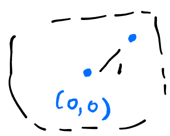
$FM_k := \overline{\Big\{$ 

$\Big\}}_{/\text{Möbius}} = \overline{\Big\{$ 

$\Big\}}_{/\text{trans, dil.}}$.

FM_k is a $(2k-3)$-dim'l manifold w/ corners.

e.g. $FM_2 = \overline{\Big\{$ 

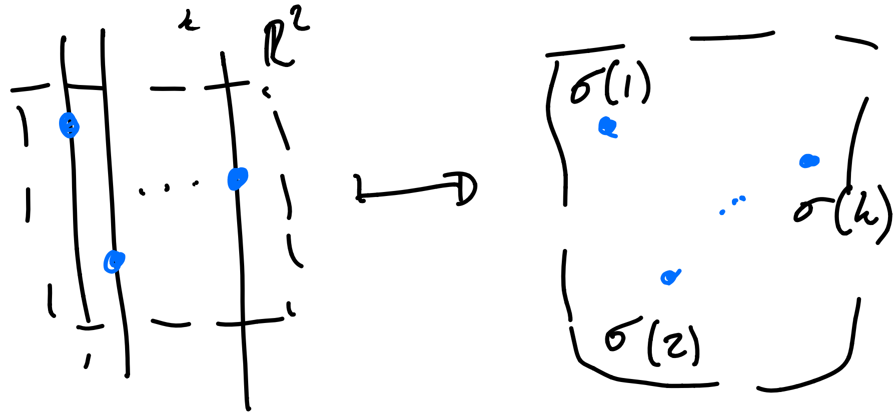
$\Big\}}_{/\sim} = \overline{\Big\{$ 

$\Big\}} = S^1$.

Fact: (FM_k) is a model for little 2-disks.

Maps between 2-associahedra, FM.

$\forall\ \sigma \in S_k$, there's a map $f_\sigma : \overline{2M}_{\underbrace{(1,\dots,1)}_{k}} \longrightarrow FM_k$



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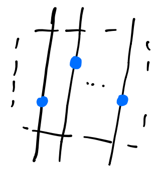
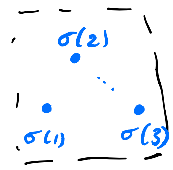




<u>Conjecture (B)</u>: There are cellular decompositions of FM_k such that:

- operadic composition $FM_k \times FM_\ell \longrightarrow FM_{k+\ell-1}$ is cellular
- the maps f_σ are cellular.

Getzler-Jones, Salvatore

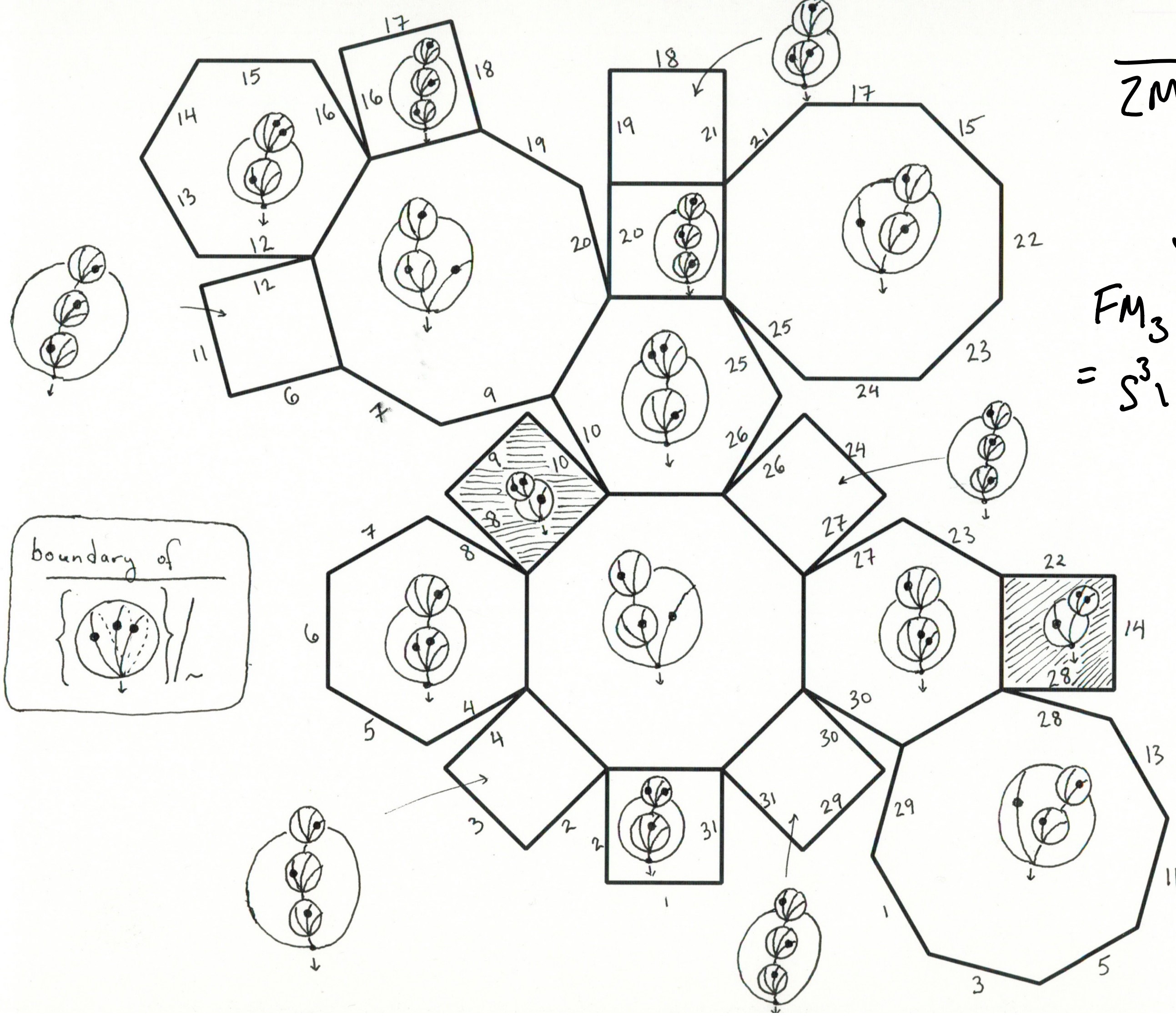
Preprint out this fall that addresses this conjecture.

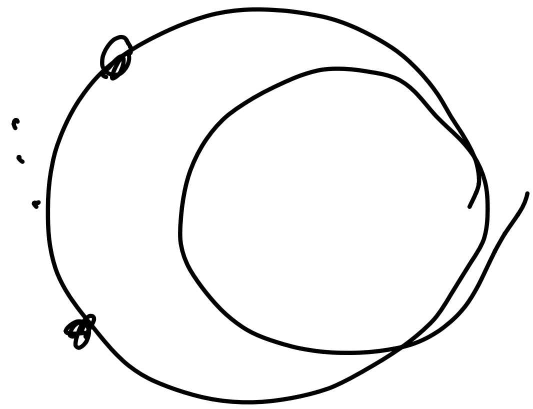
$\overline{ZM}_{111}$

$\downarrow$

FM_3

$= S^3 \setminus$ nbhd of 3-cpt link





(diamond): $\forall\ F \subset G,\ d(F) = d(G) - 2,$

$\#(F, G) = 2$

(strongly connected): $\forall\ F \subset G,\ d(G) - d(F) \geq 2,$

then (F, G) is connected.